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PRODUCTS ON THE CHOW RINGS FOR TORIC VARIETIES

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1. Introduction

Toric variety is a normal algebraic variety containing algebraic torus T_N as an open dense subset with an algebraic action of T_N which is an extension of the group law of T_N. A toric variety can be described in terms of a certain collection, which is called a fan, of cones. From this fact, the properties of a toric variety have strong connection with the combinatorial structure of the corresponding fan and the relations among the generators. That is, we can translate the difficult algebro-geometric properties of toric varieties into very simple properties about the combinatorics of cones in affine spaces over the reals.

Let $X := T_N\mathrm{emb}(\Delta)$ be the toric variety corresponding to a simplicial fan Δ. In this case, we define the Chow ring $A(N,\Delta)$ as the Stanley-Reisner ring $\mathrm{SR}(N,\Delta)$ (cf. [10]) of Δ modulo the linear equivalence relation. We see that for any $0 \le p \le r$ the homogeneous part $A^p(N,\Delta)$ of degree p of the Chow ring $A(N,\Delta)$ is generated over $\mathbf{Q}$ by the equivalence classes $v(\sigma)$ of the elements in $\mathrm{SR}(N,\Delta)$ corresponding to $\sigma \in \Delta(p)$. The product satisfies

$$v(\sigma)\cdot v(\sigma') = v(\sigma+\sigma') \quad \text{whenever} \quad \sigma+\sigma' \in \Delta \quad \text{and} \quad \sigma \cap \sigma' = \{0\}.$$

Let N' be a free $\mathbf{Z}$-module of rank r', and Δ' a fan for N'. Let $\phi : (N',\Delta') \to (N,\Delta)$ be a map of fans. Then ϕ gives rise to an equivariant holomorphic map $\phi_v : T_{N'}\mathrm{emb}(\Delta') \to T_N\mathrm{emb}(\Delta)$ between toric varieties. If the corresponding map ϕ_v is a *proper* map, then there exists a push-forward homomorphism $\phi_* : A(N',\Delta') \to A(N,\Delta)$

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between the Chow rings (cf. [2]). We have described ϕ_* explicitly in terms of the combinatorial structure of the fans Δ, Δ' and a map ϕ of fans, whenever ϕ has *finite cokernel.*

We have described the pull-back homomorphism $\phi^* : A(N, \Delta) \to A(N', \Delta')$ explicitly for an *arbitrary* map ϕ of fans. If $\phi : (N', \Delta') \to (N, \Delta)$ is a map of fans with finite cokernel, then we can prove directly that the induced homomorphisms ϕ_* and ϕ^* satisfy the *projection formula.*

In this paper, we express the exterior product and the action on the Chow rings for toric varieties in terms of corresponding simplicial fans, and state some properties of them.

2. Chow Ring

Now we introduce some basic definitions which are used throughout this paper. Let N be a free $\mathbf{Z}$-module of rank r over the ring $\mathbf{Z}$ of integers, and denote by $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ its dual $\mathbf{Z}$-module with the canonical bilinear pairing $\langle\ ,\ \rangle : M \times N \longrightarrow \mathbf{Z}$. We denote the scalar extensions of N and M to the field $\mathbf{R}$ of real numbers by $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$ and $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$, respectively.

A subset σ of $N_{\mathbf{R}}$ is called a *rational convex polyhedral cone* (or a *cone,* for short), if there exist a finite number of elements $n_1, n_2, \ldots, n_s$ in N such that $\sigma = \mathbf{R}_{\geq 0} n_1 + \mathbf{R}_{\geq 0} n_2 + \ldots + \mathbf{R}_{\geq 0} n_s$ where we denote by $\mathbf{R}_{\geq 0}$ the set of nonnegative real numbers. σ is said to be *strongly convex* if it contains no nontrivial subspace of $\mathbf{R}$, that is, $\sigma \cap (-\sigma) = \{0\}$.

A subset τ of σ is called a *face* and denoted by $\tau \prec \sigma$, if

$$\tau = \sigma \cap \{m_0\}^{\perp} := \{y \in \sigma \mid \langle m_0, y \rangle = 0\}$$

for an $m_0 \in \sigma^{\vee}$, where $\sigma^{\vee} := \{x \in M_{\mathbf{R}} \mid \langle x, y \rangle \geq 0 \text{ for all } y \in \sigma\}$ is the *dual* cone of σ.

DEFINITION. A finite collection Δ of strongly convex cones in $N_{\mathbf{R}}$ is called a *fan* if it satisfies the following conditions:

(i) Every face of any $\sigma \in \Delta$ is contained in Δ.

(ii) For any $\sigma, \sigma' \in \Delta$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

A cone σ is said to be *simplicial* if there exist $\mathbf{R}$-linearly independent elements $\{n_1, n_2, \ldots, n_s\}$ in N such that σ can be expressed as

$$\sigma = \mathbf{R}_{\geq 0} n_1 + \cdots + \mathbf{R}_{\geq 0} n_s.$$

We say that a fan Δ is *simplicial* if every cone $\sigma \in \Delta$ is simplicial. A fan Δ is said to be *complete* if $|\Delta| := \bigcup_{\sigma \in \Delta} \sigma = N_{\mathbf{R}}$.

If a fan Δ is given, then there exists a toric variety $X := T_N \mathrm{emb}(\Delta)$ determined by Δ over the field $\mathbf{C}$ of complex numbers. For the precise definition of toric varieties, see [1], [3] and [8].

Let Δ be a simplicial fan for $N \cong \mathbf{Z}^r$, which may not be complete. From now, we define the Chow ring $A(N, \Delta)$ of a toric variety using the corespomding simplicial fan.

Introduce the polynomial ring $S(\Delta)$ over $\mathbf{Q}$ in the variables $\{x(\rho) \mid \rho \in \Delta(1)\}$. We can regard this ring as a graded $\mathbf{Q}$-algebra by letting $\deg x(\rho) = 1$ for any $\rho \in \Delta(1)$. Let I be the ideal in $S(\Delta)$ generated by the set

$$\left\{ x(\rho_1)x(\rho_2)\cdots x(\rho_s) \,\middle|\, \rho_1, \ldots, \rho_s \in \Delta(1) \text{ distinct and } \rho_1 + \cdots + \rho_s \notin \Delta \right\}.$$

Then the residue ring $\mathrm{SR}(N, \Delta) := S(\Delta)/I$ is the *Stanley-Reisner ring* (or *face ring* in [10]) of a fan Δ.

On the other hand, we define another ideal J in $S(\Delta)$ to be the one generated by the set

$$\left\{ \theta(m) := \sum_{\rho \in \Delta(1)} \langle m, n(\rho) \rangle x(\rho) \,\middle|\, m \in M \right\}.$$

DEFINITION. (cf. [5] and [9]) In the notation above, we define the *Chow ring* over $\mathbf{Q}$ for a *simplicial* fan Δ to be the ring

$$A(N, \Delta) := S(\Delta) \Big/ (I + J).$$

We simply write $A(\Delta)$ if there is no confusion. We denote by $A^k(\Delta)$ its homogeneous part of degree k.

Let us denote by $v(\rho) \in A^1(\Delta)$ the image in $A(\Delta)$ of $x(\rho)$ for $\rho \in \Delta(1)$. Since Δ is assumed to be simplicial, each $\sigma \in \Delta(k)$ can be expressed as $\sigma = \rho_1 + \cdots + \rho_k$ for distinct $\rho_1, \ldots, \rho_k \in \Delta(1)$. In this case, we denote by $v(\sigma) \in A^k(\Delta)$ the image in $A(\Delta)$ of the monomial $x(\rho_1)x(\rho_2)\cdots x(\rho_k) \in S(\Delta)$.

For pairs $\sigma, \sigma' \in \Delta$, we have

$$v(\sigma) \cdot v(\sigma') = \begin{cases} 0 & \text{if } \sigma + \sigma' \notin \Delta \\ v(\sigma + \sigma') & \text{if } \sigma \cap \sigma' = \{0\} \text{ and } \sigma + \sigma' \in \Delta. \end{cases}$$

REMARK. In preparing this paper, Fulton and Sturmfels sent us their preprint [4]. They described the ring structure of the Chow ring $A(X)$ of a complete toric variety X in terms of Minkowski weights on its fan Δ.

Now, we define the pull-back homomorphism and the push-forward homomorphism induced by a limited class of maps of fans.

DEFINITION. Let (N, Δ) and (N', Δ') be two fans for $N \cong \mathbf{Z}^r$ and $N' \cong \mathbf{Z}^{r'}$. A *map of fans* $\phi : (N', \Delta') \to (N, \Delta)$ is a $\mathbf{Z}$-linear homomorphism $\phi : N' \to N$ whose scalar extension $\phi : N'_{\mathbf{R}} \to N_{\mathbf{R}}$ satisfies the following property: For each $\sigma' \in \Delta'$ there exists $\sigma \in \Delta$ such that $\phi(\sigma') \subset \sigma$.

From now on, we assume that Δ and Δ' are two *simplicial* fans for $N \cong \mathbf{Z}^r$ and $N' \cong \mathbf{Z}^{r'}$. We also assume that $\phi : (N', \Delta') \to (N, \Delta)$ is a map of fans.

Then a map of fans $\phi : (N', \Delta') \to (N, \Delta)$ gives rise to a pull-back homomorphism $\phi^* : A(N, \Delta) \longrightarrow A(N', \Delta')$ which is a graded $\mathbf{Q}$-algebra homomorphism. That is, For each $\rho' \in \Delta'(1)$, there exists a unique cone

$$\sigma_{\rho'} := \rho_1 + \cdots + \rho_s \in \Delta \quad \text{for some} \quad \rho_1, \ldots, \rho_s \in \Delta(1),$$

which contains $\phi(n'(\rho'))$ in its relative interior, where $n'(\rho') \in N'$ is the unique primitive element contained in $\rho' \in \Delta'(1)$. Thus,

$$\phi(n'(\rho')) = c(\rho', \rho_1)n(\rho_1) + \cdots + c(\rho', \rho_s)n(\rho_s)$$

$$= \sum_{\rho \in \Delta(1);\, \rho \prec \sigma_{\rho'}} c(\rho', \rho)n(\rho)$$

for some $c(\rho', \rho) > 0$. In this notation, we define the pull-back homomorphism

$$\phi^*(v(\rho)) = \sum_{\rho' \in \Delta'(1), \rho \prec \sigma_{\rho'}} c(\rho', \rho) v'(\rho').$$

We have $\psi^* \circ \phi^* = (\phi \circ \psi)^*$.

On the other hand, let $\phi : (N', \Delta') \to (N, \Delta)$ be a proper map of fans, which has finite cokernel. Then ϕ gives rise to the push-forward **Q**-linear map

$$\phi_* : A^p(N', \Delta') \longrightarrow A^{p-(r'-r)}(N, \Delta)$$

for all p, using multiplicity(cf. [3] and [6]) of a cone. Moreover, the push-forward homomorphisms satisfy $\phi_* \circ \psi_* = (\phi \circ \psi)_*$.

Furthermore, ϕ^* and ϕ_* satisfy the *projection formula* , that is, for all $\omega \in A(\Delta)$ and $\omega' \in A(\Delta')$, $\phi_*(\phi^*(\omega) \cdot \omega') = \omega \cdot \phi_*(\omega')$.

For the precise definition and more properties see [9].

3. Products on the Chow rings

In this section, we define the exterior product and the action on the Chow rings for toric varieties, using the homomorphisms defined in the last section.

We see that the product of the toric varieties has the following properties:

PROPOSITION 3.1. (cf. [7, Proposition 7.2]) *Let Δ', Δ'' be fans for N', N'', respectively and*

$$\Delta' \times \Delta'' := \{\sigma = \sigma' \times \sigma'' \mid \sigma' \in \Delta', \sigma'' \in \Delta''\}$$

be a fan for $N' \times N''$. Then we have

$$T_{N'}\mathrm{emb}(\Delta') \times T_{N''}\mathrm{emb}(\Delta'') = T_{N' \times N''}\mathrm{emb}(\Delta' \times \Delta'').$$

THEOREM 3.2. *Let Δ, Δ' be simplicial fans for N, N', respectively. Then we have the exterior product on the Chow rings for toric varieties*

$$A^k(\Delta) \otimes A^l(\Delta') \quad \dashrightarrow \quad A^{k+l}(\Delta \times \Delta'),$$

which is associative and

$$\bigoplus_{k+l=m} A^k(\Delta) \otimes A^l(\Delta') \quad \longrightarrow \quad A^m(\Delta \times \Delta')$$

becomes an isomorphism.

Proof. Recall the definition of the Chow ring. Let us denote the generator of $S(\Delta)$ (resp. $S(\Delta')$, resp. $S(\Delta \times \Delta')$) by $x(\rho)$(resp. $y(\rho')$, resp. $z(\rho'')$), and the one of $A(\Delta)$ (resp. $A(\Delta')$, resp. $A(\Delta \times \Delta')$) by $v(\rho)$ (resp. $v'(\rho')$, resp. $v''(\rho'')$). Also we denote by J'(resp. J'') in $S(\Delta')$ (resp. $S(\Delta \times \Delta')$) the one corresponding to J in $S(\Delta)$.

Define $\phi : S(\Delta) \otimes S(\Delta') \to S(\Delta \times \Delta')$ by

$$\begin{aligned}\phi(x(\rho_1) \cdots x(\rho_k) \times y(\rho'_1) \cdots y(\rho'_l))\\ := z(\rho_1 \times \{0\}) \cdots z(\rho_k \times \{0\}) z(\{0\} \times \rho'_1) \cdots z(\{0\} \times \rho'_l),\end{aligned}$$

then for any distinct $\rho_1, ..., \rho_k \in \Delta(1)$, the image of ϕ is 0 whenever $\rho_1 + \cdots + \rho_k \notin \Delta$, and similarly, for any distinct $\rho'_1 ..., \rho'_l \in \Delta'(1)$, the image of ϕ is 0 whenever $\rho'_1 + \cdots + \rho'_l \notin \Delta'$. Furthermore, for any element $\sum_{\rho \in \Delta(1)} \langle m, n(\rho) \rangle x(\rho)$ in J,

$$\begin{aligned}\phi\left(\sum_{\rho \in \Delta(1)} \langle m, n(\rho) \rangle x(\rho) \times 1\right)\\ = \sum_{\rho \in \Delta(1)} \langle m, n(\rho) \rangle z(\rho \times \{0\})\\ = \sum_{\rho \in \Delta(1)} \langle m \times 0, n(\rho) \times 0 \rangle z(\rho \times \{0\}) \in J'',\end{aligned}$$

and similarly for any element $\sum_{\rho' \in \Delta'(1)} \langle m', n'(\rho') \rangle y(\rho') \in J'$, we have

$$\phi\left(1 \times \sum_{\rho' \in \Delta'(1)} \langle m', n'(\rho') \rangle y(\rho')\right) \in J''.$$

Hence ϕ induces exterior product

$$A^k(\Delta) \otimes A^l(\Delta') \longrightarrow A^{k+l}(\Delta \times \Delta').$$

Note that

$$v(\sigma) \times v'(\sigma') = v''(\sigma \times \sigma')$$

for any $\sigma \in \Delta, \sigma' \in \Delta'$, by definition. It is clear that this exterior product is associative. Since the cone Σ'' corresponding to $v''(\sigma'') \in A^m(\Delta'')$ has the form $\sigma'' = \sigma \times \sigma', \sigma \in \Delta(k), \sigma' \in \Delta'(l), k+l=m$, we can show that

$$\bigoplus_{k+l=m} A^k(\Delta) \otimes A^l(\Delta') \longrightarrow A^m(\Delta \times \Delta')$$

becomes isomorphism. q.e.d.

THEOREM 3.3. *Let $\Delta_i, \Delta'_i, i = 1, 2$, be simplicial fans for N_i, N'_i, respectively and $f : (N'_1, \Delta'_1) \to (N_1, \Delta_1)$, $g : (N'_2, \Delta'_2) \to (N_2, \Delta_2)$ maps of fans. Then we have the following for the map of fans*

$$f \times g : (N'_1 \times N'_2, \Delta'_1 \times \Delta'_2) \to (N_1 \times N_2, \Delta_1 \times \Delta_2).$$

(1) *For any $\alpha \in A(\Delta_1), \beta \in A(\Delta_2)$, we have*

$$(f \times g)^*(\alpha \times \beta) = f^*(\alpha) \times g^*(\beta).$$

(2) *If maps of fans f, g have finite cokernel, then for any $\alpha' \in A(\Delta'_1), \beta' \in A(\Delta'_2)$, we have*

$$(f \times g)_*(\alpha' \times \beta') = f_*(\alpha') \times g_*(\beta').$$

Proof. (1) Since $f \times g = (f \times \mathrm{id}_{\Delta_2}) \circ (\mathrm{id}_{\Delta'_1} \times g)$, we have $(f \times g)^* = (\mathrm{id}_{\Delta'_1} \times g)^* \circ (f \times \mathrm{id}_{\Delta_2})^*$. Hence we have

$$\begin{aligned}
&(f \times \mathrm{id}_{\Delta_2})^*(v_1(\rho_1) \times v_2(\rho_2)) \\
&= \left(\sum_{\rho'_1 \in \Delta'_1(1), \rho_1 \prec \sigma_{\rho'_1}} c(\rho'_1, \rho_1) v'_1(\rho'_1) \right) \times v_2(\rho_2) \\
&= \sum_{\rho'_1 \in \Delta'_1(1), \rho_1 \prec \sigma_{\rho'_1}} c(\rho'_1, \rho_1)(v'_1(\rho'_1) \times v_2(\rho_2))
\end{aligned}$$

for any $\rho_i \in \Delta_i(1), i = 1, 2$, and it induces

$$
\begin{aligned}
&(\mathrm{id}_{\Delta_1'} \times g)^* \circ (f \times \mathrm{id}_{\Delta_2})^*(v_1(\rho_1) \times v_2(\rho_2)) \\
&= \sum_{\rho_1' \in \Delta_1'(1), \rho_1 \prec \sigma_{\rho_1'}} c(\rho_1', \rho_1) \\
&\qquad \cdot \left[v_1'(\rho_1') \times \left(\sum_{\rho_2' \in \Delta_2'(1), \rho_2 \prec \sigma_{\rho_2'}} c(\rho_2', \rho_2) v_2'(\rho_2') \right) \right] \\
&= \sum_{\rho_1' \in \Delta_1'(1), \rho_1 \prec \sigma_{\rho_1'}} c(\rho_1', \rho_1) \sum_{\rho_2' \in \Delta_2'(1), \rho_2 \prec \sigma_{\rho_2'}} c(\rho_2', \rho_2) (v_1'(\rho_1') \times v_2'(\rho_2')) \\
&= \left(\sum_{\rho_1' \in \Delta_1'(1), \rho_1 \prec \sigma_{\rho_1'}} c(\rho_1', \rho_1) v_1'(\rho_1') \right) \\
&\qquad \times \left(\sum_{\rho_2' \in \Delta_2'(1), \rho_2 \prec \sigma_{\rho_2'}} c(\rho_2', \rho_2) v_2'(\rho_2') \right) \\
&= f^*(v_1(\rho_1)) \times g^*(v_2(\rho_2)).
\end{aligned}
$$

(2) Similarily, we can show that $(f \times g)_*(\alpha' \times \beta') = f_*(\alpha') \times g_*(\beta')$ by using $(f \times g)_* = (f \times \mathrm{id}_{\Delta_2})_* \circ (\mathrm{id}_{\Delta_1'} \times g)_*$. q.e.d.

DEFINITION. Let $f : (N', \Delta') \to (N, \Delta)$ be a map of fans between simplicial fans Δ, Δ' for N, N', respectively. We define the *action* on the Chow rings as follows:

$$
\begin{aligned}
A^k(\Delta') \otimes A^l(\Delta) &\longrightarrow \quad A^{k+l}(\Delta') \\
\alpha \otimes \beta \quad &\longmapsto \alpha \cdot_f \beta := \alpha \cdot f^*(\beta).
\end{aligned}
$$

THEOREM 3.4. *Let $\Delta, \Delta', \Delta''$ be simplicial fans for N, N', N'', respectively and*

$$
(N'', \Delta'') \xrightarrow{\ f\ } (N', \Delta') \xrightarrow{\ g\ } (N, \Delta)
$$

be maps of fans between them.

(1) *The action is associative, that is,*

$$\alpha \cdot_f (\beta \cdot_g \gamma) = (\alpha \cdot_f \beta) \cdot_{gf} \gamma,$$

for any $\alpha \in A(\Delta''), \beta \in A(\Delta')$, *and* $\gamma \in A(\Delta)$.

(2) *If the map of fans* $f : (N'', \Delta'') \to (N', \Delta')$ *is proper with finite cokernel, then it satisfies the* projection formula :

$$f_*(\alpha \cdot_{gf} \gamma) = f_*(\alpha) \cdot_g \gamma,$$

for any $\alpha \in A(\Delta'')$ *and* $\gamma \in A(\Delta)$.

Proof. (1) For any $\alpha \in A(\Delta''), \beta \in A(\Delta')$, and $\gamma \in A(\Delta)$, we have

$$\begin{aligned} \alpha \cdot_f (\beta \cdot_g \gamma) &= \alpha \cdot_f (\beta \cdot g^*(\gamma)) \\ &= \alpha \cdot f^*(\beta \cdot g^*(\gamma)) \\ &= \alpha \cdot f^*(\beta) \cdot f^*(g^*(\gamma)) \\ &= (\alpha \cdot f^*(\beta)) \cdot (gf)^*(\gamma) \\ &= (\alpha \cdot_f \beta) \cdot_{gf} \gamma. \end{aligned}$$

(2) Since the map of fans $f : (N'', \Delta'') \to (N', \Delta')$ satisfies the projection formular

$$f_*(f^*(\beta) \cdot \alpha) = \beta \cdot f_*(\alpha)$$

for any $\alpha \in A(\Delta'')$ and $\beta \in A(\Delta')$, it is clear that

$$f_*(\alpha \cdot_{gf} \gamma) = f_*(\alpha) \cdot_g \gamma.$$

q.e.d.

THEOREM 3.5. *Let* $\Delta, \Delta_i, \Delta_i', 1 \leq i \leq m$ *be simplicial fans for* N, N_i, N_i', *respectively.*

(1) *If* $f_i : (N_i', \Delta_i') \to (N_i, \Delta), 1 \leq i \leq m$ *are maps of fans, then we have*

$$(\alpha_1 \times \cdots \times \alpha_m) \cdot_{(f_1 \times \cdots \times f_m)} (\beta_1 \times \cdots \times \beta_m) = (\alpha_1 \cdot_{f_1} \beta_1) \times \cdots \times (\alpha_m \cdot_{f_m} \beta_m),$$

for any $\alpha_i \in A(\Delta_i')$ *and* $\beta_i \in A(\Delta_i)$.

(2) *If* $g_1 : \Delta \to \Delta_1, g_2 : \Delta \to \Delta_2$ *are maps of fans, then we have*

$$(\alpha \cdot_{g_1} \beta_1) \cdot_{g_2} \beta_2 = (\alpha \cdot_{g_2} \beta_2) \cdot_{g_1} \beta_1,$$

for any $\alpha \in A(\Delta)$ *and* $\beta_i \in A(\Delta_i')$.

Proof. (2) may be easily proved.

Now we prove (1) in case of $m = 2$. By Theorem 3.3, we have

$$(f_1 \times f_2)^*(\beta_1 \times \beta_2) = f_1^*(\beta_1) \times f_2^*(\beta_2),$$

and it follows that

$$\begin{aligned}(\alpha_1 \times \alpha_2) \cdot_{(f_1 \times f_2)} (\beta_1 \times \beta_2) &= (\alpha_1 \times \alpha_2) \cdot (f_1 \times f_2)^*(\beta_1 \times \beta_2) \\ &= (\alpha_1 \times \alpha_2) \cdot (f_1^*(\beta_1) \times f_2^*(\beta_2)) \\ &= (\alpha_1 \cdot f_1^*(\beta_1)) \times (\alpha_2 \cdot f_2^*(\beta_2)) \\ &= (\alpha_1 \cdot_{f_1} \beta_1) \times (\alpha_2 \cdot_{f_2} \beta_2).\end{aligned}$$

q.e.d.

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