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ENUMERATION OF NSEW–PATHS IN RESTRICTED PLANES

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1. Introduction

A path g in the plane $\mathbb{R}^2$ is the sequence of the points $(t_0, t_1, ..., t_n)$, with coordinates in $\mathbb{Z}^2$. The point t_0 is the *starting point* and the point t_n is the *arriving point*. An *elementary step* of g is a couple (t_i, t_{i+1}), $0 \le i \le n-1$. We denote the length of the path g by $|g| = n$.

An *NSEW-path* is a path which consists of four elementary steps :

$$\begin{aligned}
&\text{East step } \ (t_i = (x_i, y_i), \text{ and } t_{i+1} = (x_i + 1, y_i)),\\
&\text{West step } \ (t_i = (x_i, y_i), \text{ and } t_{i+1} = (x_i - 1, y_i)),\\
&\text{North step } \ (t_i = (x_i, y_i), \text{ and } t_{i+1} = (x_i, y_i + 1)),\\
&\text{South step } \ (t_i = (x_i, y_i), \text{ and } t_{i+1} = (x_i, y_i - 1)).
\end{aligned}$$

A path $g = (t_0, t_1, ..., t_{2n})$ is called a *Dyck path* of the length $2n$ if the t_i's are never under the x axis and $t_0 = (m, 0)$, $t_{2n} = (2n + m, 0)$, for an integer m, and the elementary step (t_i, t_{i+1}) is either North-East step $(t_i = (x_i, y_i)$, and $t_{i+1} = (x_i + 1, y_i + 1))$, or South-East step $(t_i = (x_i, y_i)$, and $t_{i+1} = (x_i + 1, y_i - 1))$, $1 \le i \le 2n - 1$.

D. André (1887) has enumerated *the minimal paths* going from (0,0) to any point in the restricted plane by a line $y = x + c$ $(c \ge 1)$, using the *reflection principle*. The reflection principle contributed to enumerate various classes of paths, and by using this reflection principle, the authors can obtain many important results and the paths theory become an important domain in combinatorics.

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The main results for NSEW-paths are as follows. Détemple and Robertson [4], [5] have given a formula for the number of NSEW-paths in a plane $\mathbb{R}^2$ or in restricted planes.

Guy, Krattenthaler and Sagan [12] have given a formula for the number of NSEW-paths going from a point (a, b) to a point (a', b') without crossing the x axis or y axis.

The enumeration of paths going from the origin to a point (a, b) in a triangle $\{(x, y)|x \geq 0, y \geq 0, x + y \leq m\}$, is discussed by Flajolet [6]. Arques [1] has given a formula for the number of NSEW-paths going from the origin to the origin in the triangle.

Gouyou-Beauchamps [10] has enumerated NSEW-paths going from (0,0) to a point on the x axis in the eighth plane, $\{(x, y)|x \geq y \geq 0\}$. He has shown that the formula of NSEW-paths enumerate also the standard Young tableaux with height 4 or 5 [11].

In this paper, we improve the results of Gouyou-Beauchamps [10] and we find that the number of NSEW-paths going from (0,0) to any point (a,b) in the eighth plane is equal to

$$\frac{(b+1)(a+2)(a-b+1)(a+b+3)n!(n+2)!}{(\frac{n-a-b}{2})!(\frac{n-a+b}{2}+1)!(\frac{n+a-b}{2}+2)!(\frac{n+a+b}{2}+3)!}$$

where $n \geq a + b$ and $n \equiv a + b (mod\ 2)$.

2. Preliminary notion

Let $N_{n,a,b}$ be the set of all NSEW-paths of length n going from (0,0) to (a, b) in the plane. Let $|N_{n,a,b}|$ be the cardinality of $N_{n,a,b}$. An NSEW-path of $N_{n,a,b}$ is said to be *sub-diagonal* if it never passes over the line $y = x$, and if the coordinates of summits of the NSEW-path are positive integers. For $N_{n,a,b}$ of the sub-diagonal paths, we assume $0 \leq b \leq a$.

By a *minimal sub-diagonal* path, we mean a path consisting of two elementary steps: East and North, which never passes over the line $y = x$, and the coordinates of whose summits are positive integers. In this case, the path can pass over a summit at most one time.

The basic result for NSEW-paths is the following. Detemple and Robertson [4] have given that the cardinality of $N_{n,a,b}$ is equal to

$$\binom{n}{\frac{n-a-b}{2}}\binom{n}{\frac{n+a-b}{2}}$$

where $n-a-b$ is even (if $n-a-b$ is odd, $|N_{n,a,b}|=0$). Arques [1] has given another proof for the same result by using the generating function of NSEW-paths.

Let $H_{n,a,b}$ (resp. $Q_{n,a,b}$) be the set of all NSEW-paths of length n, which are going from $(0,0)$ to (a,b), and lying in the half plane $\{(x,y)|x\geq 0\}$ (resp. the quarter of plane $\{(x,y)|x\geq 0, y\geq 0\}$). We can obtain the cardinalities of $H_{n,a,b}$ and $Q_{n,a,b}$ by using the reflection principle [3] :

$$|H_{n,a,b}|=|N_{n,a,b}|-|N_{n,-a-2,b}|=\frac{a+1}{n+1}\binom{n+1}{\frac{n-a-b}{2}}\binom{n+1}{\frac{n-a+b}{2}}.$$

$$\begin{aligned}|Q_{n,a,b}| &=|H_{n,a,b}|-|H_{n,a,-b-2}|\\ &=|N_{n,a,b}|-|N_{n,-a-2,b}|-|N_{n,a,-b-2}|+|N_{n,-a-2,-b-2}|\\ &=\frac{(a+1)(b+1)}{(n+1)(n+2)}\binom{n+2}{\frac{n-a-b}{2}}\binom{n+2}{\frac{n-a+b}{2}}.\end{aligned}$$

In the case $a=b=0$, Arques [1] has given the following formulas :

$$\begin{aligned}H_{2n,0,0} &= \binom{2n+1}{n}C_n,\\ Q_{2n,0,0} &= C_nC_{n+1},\end{aligned}$$

where C_n is the n^{th} Catalan number.

Let $N_{n,a,b;y=x+k}$ denote the set of all NSEW-paths which do not touch the line $y=x+k$, $0<k\leq(n+a-b)/2$, and let $N_{n,a,b;y=x/k}$ be the set of all NSEW-paths which do not touch the line of the form $y=x/k$, $a>kb$.

Detemple and Robertson [3] have obtained the following two formulas :

$$\begin{aligned}|N_{n,a,b;y=x+k}| &= |N_{n,a,b}| - |N_{n,a+k,b-k}| \\ &= \binom{n}{\frac{n-a-b}{2}}\left(\binom{n}{\frac{n+a-b}{2}} - \binom{n}{\frac{n+a-b-2k}{2}}\right).\end{aligned}$$

$$|N_{n,a,b;y=x/k}| = \frac{a-kb}{n}\binom{n}{\frac{n-a-b}{2}}\binom{n}{\frac{n-a-b}{2}}.$$

We note that $|N_{n,a+k,b-k}| = 0$ if $k > (n+a-b)/2$, and

$$|N_{n,a,b;y=x+1}| = \frac{(-a+b+1)}{(n+1)}\binom{n}{\frac{n-a-b}{2}}\binom{n+1}{\frac{n+a-b}{2}}.$$

3. Enumeration of NSEW-paths in eighth plane

We use finite sets, called alphabets. The elements of an alphabet are called letters. A word is a finite sequence of letters and an empty word will be denoted by ϵ. The set of all words over the alphabet X is denoted by X^*. The *language* L of the alphabet X is a subset of X^*.

The length of a word f, denoted by $|f|$, is the number of letters of f. For a letter x, $|f|_x$ denotes the number of x in f. A word f' is a left factor of a word f if there exists a word $f'' \in X^*$ such that $f = f'f''$.

Let alphabets be $Z = \{x, \overline{x}, y, \overline{y}\}$ and $A = \{a, \overline{a}\}$. Consider the two morphisms δ_x and δ_y of Z^* into N defined by

$$\begin{aligned}&\delta_x(x) = 1,\ \delta_x(\overline{x}) = -1,\\ &\delta_x(y) = \delta_x(\overline{y}) = 0, \text{ and}\\ &\delta_y(y) = 1,\ \delta_y(\overline{y}) = -1,\\ &\delta_y(x) = \delta_y(\overline{x}) = 0;\end{aligned}$$

and the morphism β from A^* to $\mathbf{N}$ by

$$\beta(a) = 1,\ \beta(\overline{a}) = -1.$$

DEFINITION 1. The *Dyck language*, denoted by D, is the set of words f of A^* satisfying the following conditions :

(i) $\beta(f) = 0$,

(ii) For any left factor f' of f, $\beta(f') \geq 0$.

The words of the Dyck language is called Dyck words. The set of all left factors (of words of D) whose length is l and whose image under β is p, is denoted by $F_{l,p}$, where l and p are the same parity.

If we code the East step by a and the North step by $\overline{a}$, then Dyck words code the minimal sub-diagonal paths which start the origin and arrive on the line $y = x$ [3]. If we code the North-East steps by a and South-East steps by $\overline{a}$, then the Dyck words code the Dyck paths. The words of $F_{l,p}$ code also the left factor of Dyck path of length l which arrive at height p, with l and p having the same parity.

We have [3]

$$|D \cap A^{2n}| = C_n = \frac{1}{n+1}\binom{2n}{n},$$

where C_n is the n^{th} Catalan number, and

$$|F_{l,p}| = (p+1)\frac{l!}{(\frac{l-p}{2})!(\frac{l+p}{2}+1)!}.$$

DEFINITION 2. Let $S_{n,p,q}$ be the language which consists of words f of Z^* satisfying the following conditions :

(i) $\delta_x(f) = p$, $\delta_y(f) = q$,

(ii) for all f', left factor of f, $\delta_x(f') \geq \delta_y(f') \geq 0$,

(iii) $|f| = n$.

If we code East (resp. West) by x (resp. $\overline{x}$) and North (resp. South) by y (resp. $\overline{y}$), then the language $S_{n,p,q}$ code the sub-diagonal of length n going from (0,0) to (p, q).

DEFINITION 3. The pair (g, h) of $F_{n,q} \times F_{n,p}$, $0 \leq q \leq p \leq n$, $(n, p$ and q being the same parity), is called noncrossing words if $\beta(h') \leq \beta(g')$, for all left factors h' of h and g' of g such that $|h'| = |g'|$.

Let $R_{n,p,q}$ be the set of pairs (g, h) which do not cross the words of $F_{n,p-q} \times F_{n,p+q}$ (n and $p + q$ of the same parity).

LEMMA 1. *There is a bijection between $S_{n,p,q}$ and $R_{n,p,q}$.*

Proof. We define the morphism I from Z^* to $Z^* \times Z^*$ in the following way : $I(x) = (a,a)$, $I(\overline{x}) = (\overline{a},\overline{a})$, $I(y) = (\overline{a},a)$, $I(\overline{y}) = (a,\overline{a})$.

Let f be a word of $S_{n,p,q}$. Put $I(f) = (g,h)$. We find that $|f| = |g| = |h|$, by the construction of morphism I. For left factors f', g', h' of f, g and h respectively, we have $\beta(g') = \delta_x(f') - \delta_y(f')$ and $\delta(h') = \delta_x(f') + \delta_y(f')$.

We have then $\beta(h') \geq \beta(g') \geq 0$, $\beta(g) = p - q$ and $\beta(h) = p + q$. Therefore, (g,h) is a pair of noncrossing words.

Conversely, from a pair of words of $R_{n,p,q}$, we can construct a word f of length n by the inverse operation. Let f', g' and h' be the left factors of f, g and h. It is straightforward to see that $\delta_x(f') = (\beta(h')+\beta(g'))/2$ and $\delta_y(f') = (\beta(h')-\beta(g'))/2$. Also $\delta_x(f) = p$, $\delta_y(f) = q$, and $\delta_x(f') \geq \delta_y(f') \geq 0$. □

The Lemma 1 can be viewed as a generalization of [10, Theorem 2].

DEFINITION 4. The pair (g,h) of $F_{n,p} \times F_{n+2,p+2}$ (n and p of the same parity) is a pair of *nontouching words* if we have $\beta(h') > \beta(g')$, for all left factors h' of h and g' of g such that $|h'| = |g'| + 2$.

Let $T_{n,p,q}$ be the set of pairs of nontouching words of $F_{n,p-q} \times F_{n+2,p+q+2}$, $p \geq q \geq 0$,
(n, p and q of the same parity).

THEOREM 1. *The cardinality of $S_{n,p,q}$ is equal to :*

$$|S_{n,p,q}| = \frac{(q+1)(p+2)(p-q+1)(p+q+3)n!(n+2)!}{(\frac{n-p-q}{2})!(\frac{n-p+q}{2}+1)!(\frac{n+p-q}{2}+2)!(\frac{n+p+q}{2}+3)!}.$$

Proof. We note that $|T_{n,p,q}| = |R_{n,p,q}|$, because (g,h) is in $R_{n,p,q}$ if and only if (g,aah) is in $T_{n,p,q}$. For $(g,h) \in F_{n,p-q} \times F_{n+2,p+q+2}$, let $g_3, g_4, ..., g_{n+2}$ be the letters of A composing g, and $h_1, h_2, ..., h_{n+2}$ the letters of A composing h.

The paths g and h can touch, or not. In the case where they touch each other, there is an integer i ($3 \leq i \leq n+1$) such that $\beta(h_1h_2...h_i) \leq \beta(g_3g_4...g_i)$.

Let j the smallest index ($3 \leq j \leq n+1$) such that $\beta(h_1h_2...h_j) = \beta(g_3g_4..., g_j)$. We make two words g'' and h'' in the following way :

$g'' = h_1 h_2 \dots h_j g_{j+1} g_{j+2} \dots g_{n+2}$ and $h'' = g_3 g_4 \dots g_j h_{j+1} h_{j+2} \dots h_{n+2}$.

It is straightforward to see that $(g'', h'') \in F_{n+2,p-q} \times F_{n,p+q+2}$.

By the same process, we can give a correspondence of an element $(g'', h'') \in F_{n+2,p-q} \times F_{n,p+q+2}$ to $(g, h) \in F_{n,p-q} \times F_{n+2,p+q+2}$ that g and h touch each other.

In fact, for the pair of paths g'' and h'', It sufficient to invert them at the point where they meet for the first time. This correspondence is bijective.

So we have $|S_{n,p,q}| = |R_{n,p,q}| = |T_{n,p,q}|$ is equal to

$$|F_{n,p-q}||F_{n+2,p+q+2}| - |F_{n+2,p-q}||F_{n,p+q+2}|.$$

By a straightforward calculation, we can obtain the following equality :

$$|F_{n,p-q}||F_{n+2,p+q+2}| - |F_{n+2,p-q}||F_{n,p+q+2}| = \frac{(q+1)(p+2)(p-q+1)(p+q+3)n!(n+2)!}{(\frac{n-p-q}{2})!(\frac{n-p+q}{2}+1)!(\frac{n+p-q}{2}+2)!(\frac{n+p+q}{2}+3)!}. \qquad \square$$

THEOREM 2. *The number $K(n, \alpha, p, q)$ of NSEW-path going from $(0,0)$ to any point (p, q) without touching the line $x = \alpha$, for a positive integer α, in eighth plane defined by $\{(x, y)|0 \le y \le x\}$ is equal to :*

$$K(n, \alpha, p, q) = \sum_{i \ge 0, (x_{2i}+y_{2i}) \le n} (|S_{n,x_{2i},y_{2i}}| - |S_{n,x_{2i+1},y_{2i+1}}|)$$

where $x_0 = p$, $y_0 = q$, $x_1 = -p + 2\alpha$, $y_1 = q$, $x_2 = -q + 2\alpha + 1$, $y_2 = -x_1 + 2\alpha + 1$. We have $x_{2i} = -y_{2i-1} + 2^i(\alpha+1) + 2^{i-1} + \dots + 2^1 + 2^0$, $y_{2i} = -x_{2i-1} + 2^i(\alpha + 1) + 2^{i-1} + \dots + 2^1 + 2^0$, for $i \ge 2$, and $x_{2i+1} = -x_{2i} + 2^i(\alpha + 1) + 2^{i-1} + \dots + 2^1$, $y_{2i+1} = -y_{2i}$, for $i \ge 1$.

Proof. We use the reflection principle in this problem as follows. If the path touch the line $x = \alpha$ at a point (c, d), then the number of paths going from (0,0) to (p, q) through (c, d) is equal to the number of paths going from (0,0) to $(p + 2(\alpha - p), q)$ through (c, d), because $(p+2(\alpha-p), q)$ is symmetric with (p, q) about the line $x = \alpha$. As $|S_{n,p,q}|$ is the number of NSEW-paths going from (0,0) to any point (p, q) in the eighth plane defined by $\{(x,y)|0 \le y \le x\}$, we have $K(n,\alpha,p,q) = |S_{n,p,q}|$ − (the number of NSEW-paths going from (0,0) to $(p+2(\alpha-p),q)$ no touching the line $y=-x+2\alpha+1$).

As the point $(-q+2\alpha+1, -p+2\alpha+1)$ is symmetric with $(p+2(\alpha-p),q)$ w.r.t. the line $y=-x+2\alpha+1$, the number of paths going from (0,0) to $(p+2(\alpha-p),q)$ and touching the line $y=-x+2\alpha+1$ at a point (c',d') is equal to the number of paths going from (0,0) to $(-q+2\alpha+1,-p+2\alpha+1)$ through the point (c',d'). So we can obtain $K(n,\alpha,p,q) = |S_{n,p,q}| - |S_{n,p+2(\alpha-p),q}|$ + (the number of NSEW-paths going from (0,0) to $(-q+2\alpha+1,-p+2\alpha+1)$ without touching the line $x=2\alpha+2$).

If we apply this procedure successively , we can obtain the inclusion-exclusion formula :

$$K(n,\alpha,p,q) = \sum_{i\ge 0,(x_{2i}+y_{2i})\le n} (|S_{n,x_{2i},y_{2i}}| - |S_{n,x_{2i+1},y_{2i+1}}|)$$

where $x_0=p$, $y_0=q$, $x_1=-p+2\alpha$, $y_1=q$, $x_2=-q+2\alpha+1$, $y_2=-x_1+2\alpha+1$. We have $x_{2i}=-y_{2i-1}+2^i(\alpha+1)+2^{i-1}+\ldots+2^1+2^0$, $y_{2i}=-x_{2i-1}+2^i(\alpha+1)+2^{i-1}+\ldots+2^1+2^0$, for $i\ge 2$, and $x_{2i+1}=-x_{2i}+2^i(\alpha+1)+2^{i-1}+\ldots+2^1$, $y_{2i+1}=-y_{2i}$, for $i\ge 1$. □

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