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CAUCHY PROBLEMS FOR A PARTIAL DIFFERENTIAL EQUATION IN WHITE NOISE ANALYSIS

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1. Introduction

The Gross Laplacian Δ_G was introduced by Gross for a function defined on an abstract Wiener space (H, B) [1,7]. Suppose φ is a twice H-differentiable function defined on B such that $\varphi''(x)$ is a trace class operator of H for every $x \in B$. Then the Gross Laplacian $\Delta_G\varphi$ of φ is defined by

$$\Delta_G\varphi(x) = trace_H \varphi''(x).$$

If in addition $\varphi'(x) \in B^*$, then the number operator $N\varphi$ of φ is defined by

$$N\varphi(x) = -trace_H\varphi''(x) + \langle x, \varphi'(x)\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the $B - B^*$ pairing.

In [1], Gross studied the solution of the heat equation associated with the Gross Laplacian Δ_G on (H, B):

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\Delta_G u(x,t), \qquad u(x,0) = f(x).$$

In [15], Piech studied the solution of the heat equation associated with the number operator N on (H, B):

$$\frac{\partial u}{\partial t}(x,t) = -Nu(x,t), \qquad u(x,0) = f(x).$$

We note that in white noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$, the white noise measure μ is supported in the space $\mathcal{S}'_p$(see section 2) for any $p > \frac{1}{2}$.

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Thus $(L^2(\mathbb{R}), \mathcal{S}_p'(\mathbb{R}))$ is an abstract Wiener space Therefore we can define $\Delta_G\varphi$ and $N\varphi$ for functions φ defined on $\mathcal{S}_p'(\mathbb{R})$.

In [9,10], Kuo has studied the heat equation associated with the Gross Laplacian Δ_G in white noise analysis setting. In [5], Kang has studied the heat equation associated with the number operator N in white noise analysis setting. In this paper we will investigate the existence of a solution of the Cauchy problem associated with the operator $\Delta_G + N$ in white noise analysis setting.

2. Preliminaries

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of real valued rapidly decreasing smooth functions on $\mathbb{R}$. The dual space $\mathcal{S}'(\mathbb{R})$ of $\mathcal{S}(\mathbb{R})$ is the space of tempered distributions on $\mathbb{R}$. Then $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ is a Gel'fand triple.

Let $A = -\frac{d^2}{dt^2} + t^2 + 1$. Then A is the densely defined self-adjoint on $L^2(\mathbb{R})$ and $Ae_n = (2n+2)e_n$, where $\{e_n\}$ is an ONB of $L^2(\mathbb{R})$ defined by

$$e_n(u) = (-1)^n(\pi^{\frac{1}{2}}2^n n!)^{-\frac{1}{2}} e^{\frac{u^2}{2}} [\frac{d^n}{du^n} e^{-u^2}].$$

For each $p \geq 0$, define

$$|f|_p = |A^p f|_0, \quad f \in L^2(\mathbb{R}),$$

where $|\cdot|_0$ is the $L^2(\mathbb{R})$-norm. Let $\mathcal{S}_p(\mathbb{R}) = \{f \in L^2(\mathbb{R}) | \, |f|_p < \infty\}$. Then $\mathcal{S}_p(\mathbb{R})$ is a real separable Hilbert space with the norm $|\cdot|_p$ and the dual space $\mathcal{S}_p'(\mathbb{R})$ of $\mathcal{S}_p(\mathbb{R})$ is given by $\mathcal{S}_{-p}(\mathbb{R})$. Furthermore, we have $\mathcal{S}_q(\mathbb{R}) \subset \mathcal{S}_p(\mathbb{R})$ for $p < q$ and

$$\mathcal{S}(\mathbb{R}) = \bigcap_{p \geq 0} \mathcal{S}_p(\mathbb{R}), \qquad \mathcal{S}'(\mathbb{R}) = \bigcup_{p \geq 0} \mathcal{S}_{-p}(\mathbb{R}).$$

Since $\mathcal{S}(\mathbb{R})$ is a nuclear space, by the Bochner-Minlos theorem there exists a unique probability measure μ on σ-algebra $\mathcal{B}$ of Borel subsets of $\mathcal{S}'(\mathbb{R})$ such that

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle x, \xi \rangle} d\mu(x) = e^{-\frac{1}{2}|\xi|_0^2}, \quad \xi \in \mathcal{S}(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ is the $\mathcal{S}'(\mathbb{R})-\mathcal{S}(\mathbb{R})$ pairing. The triple $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ is called the white noise space.

Let (L^2) be the Hilbert space of μ-square integrable functions on the white noise space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)$ with norm $||\cdot||_0$. By the Wiener-Itô decomposition theorem [3,14], every $\phi \in (L^2)$ admits a chaos decomposition:

$$\phi = \sum_{n=0}^{\infty} I_n(f_n),$$

where $I_n(f_n)$ denotes a multiple Itô integral of order n with the kernel $f_n \in \widehat{L^2}(\mathbb{R}^n)$ (the symmetric L^2-space).

It is well-known (see [17]) that $I_n(f)(x) = \langle : x^{\otimes n} :, f \rangle$, where $: x^{\otimes n} :$ is the Wick ordering. Hence for each ϕ in (L^2), we have

$$\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle \quad f_n \in \widehat{L^2}(\mathbb{R}^n).$$

Moreover, we have

$$||\phi||_0^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2.$$

The second quantization $\Gamma(A)$ of A is densely defined on (L^2) as follows: for $\phi \in (L^2)$ with $\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle$,

$$\Gamma(A)\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, A^{\otimes n} f_n \rangle.$$

For $p \geq 0$, define

$$||\phi||_p = ||\Gamma(A)^p \phi||_0$$

and let $(S)_p = \{\phi \in (L^2); ||\phi||_p < \infty\}$. Then $(\mathcal{S})_p$ is a Hilbert space with the norm $||\cdot||_p$. For $p < 0$, we define $(\mathcal{S})_p$ as the completion of (L^2) with respect to $||\cdot||_p$. Then the dual space $(\mathcal{S})_p^*$ of $(\mathcal{S})_p$ is given by $(\mathcal{S})_{-p}$, and we have

$$(\mathcal{S})_q \subset (\mathcal{S})_p \subset (L^2) \subset (\mathcal{S})_{-p} \subset (\mathcal{S})_{-q},$$

where $q > p \geq 0$. The space $(\mathcal{S})$ of test functions is the projective limit of $\{(\mathcal{S})_p; p \geq 0\}$. The space $(S)^*$ of generalized functions (or Hida distributions) is the dual space of $(\mathcal{S})$. Thus we have a Gel'fand triple $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$ and will use the symbol $\langle\langle \cdot, \cdot \rangle\rangle$ for the $(\mathcal{S})^*$-$(\mathcal{S})$ pairing.

Let $\mathcal{G}$ be a continuous linear operator from $(\mathcal{S})$ into itself defined by

$$\mathcal{G}\phi(y) = \langle\langle : e^{-i\langle \cdot, y\rangle} :, \phi \rangle\rangle, \quad y \in E^*$$

where $: e^{-i\langle \cdot, y\rangle} : \in (\mathcal{S})^*$. Then it is known [3,12] that the adjoint $\mathcal{G}^*$ of $\mathcal{G}$ is the Fourier transform $\mathcal{F}$. For any $\phi \in (\mathcal{S})$ with $\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle$, $f_n \in \widehat{S(\mathbb{R}^n)}$, $\mathcal{G}\phi$ has the following chaos decomposition

$$(2.1) \quad \mathcal{G}\phi(y) = \sum_{n=0}^{\infty} \langle : y^{\otimes n} :, \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!2^m} (-i)^{n+2m} \tau^{\otimes m} \hat{\otimes}_{2m} f_{n+2m} \rangle.$$

And it can be shown that $\mathcal{G}^3 = \mathcal{G}^{-1}$, and for any $\phi \in (\mathcal{S})$ with $\phi(y) = \sum_{n=0}^{\infty} \langle : y^{\otimes n} :, f_n \rangle$, $f_n \in \widehat{S(\mathbb{R}^n)}$, $\mathcal{G}^{-1}\phi$ is given by

$$(2.2) \quad \mathcal{G}^{-1}\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \sum_{m=0}^{\infty} \frac{(n+2m)!}{n!m!2^m} (i)^{n+2m} \tau^{\otimes m} \hat{\otimes}_{2m} f_{n+2m} \rangle,$$

where $\tau^{\otimes m} \hat{\otimes}_{2m} f_{n+2m}(u) = \int_{\mathbb{R}^m} f(t_1, t_1, t_2, t_2, \cdots, t_m, t_m, u) dt_1 \cdots dt_m$, $u \in \mathbb{R}^n$. It also can be shown by using the duality that for any $\Phi \in (\mathcal{S})^*$ with $\Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, F_n \rangle$, $F_n \in \widehat{S'(\mathbb{R}^n)}$, $\mathcal{F}\Phi$ and $\mathcal{F}^{-1}\Phi$ are given by

$$(2.3) \qquad \mathcal{F}\Phi(y) = \sum_{n=0}^{\infty} \langle : y^{\otimes n} :, \sum_{m=0}^{[\frac{n}{2}]} \frac{(-i)^n}{m!2^m} F_{n-2m} \hat{\otimes} \tau^{\otimes m} \rangle,$$

and

$$(2.4) \qquad \mathcal{F}^{-1}\Phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \sum_{m=0}^{[\frac{n}{2}]} \frac{i^n}{m!2^m} F_{n-2m} \hat{\otimes} \tau^{\otimes m} \rangle.$$

It is known [3] that for any $\phi \in (\mathcal{S})$ with $\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, f_n \rangle$, $f_n \in \widehat{S(\mathbb{R}^n)}$, $\Delta_G \phi$ and $N\phi$ are given by

$$\Delta_G \phi(x) = \sum_{n=0}^{\infty} (n+2)(n+1) \langle : x^{\otimes n} :, \tau \hat{\otimes}_2 f_{n+2} \rangle$$

and

$$N\phi(x) = \sum_{n=0}^{\infty} n\langle : x^{\otimes n} :, f_n\rangle.$$

3. Cauchy problem associated with the operator $\Delta_{\mathrm{G}} + N$

In this section we use the $\mathcal{G}$-and Fourier transforms to investigate the existence of a solution of the following Cauchy problems:

$$(3.1) \qquad \frac{\partial}{\partial t}u(x,t) = -(\Delta_{\mathrm{G}} + N)u(x,t), \qquad u(x.0) = \phi(x),$$

where $\phi \in (\mathcal{S})$, and

$$(3.2) \qquad \frac{\partial}{\partial t}u(x,t) = -(\Delta_{\mathrm{G}}^* + N)u(x,t), \qquad u(x,0) = \Phi(x),$$

where $\Phi \in (\mathcal{S})^*$ and Δ_{G}^* is the adjoint of Δ_{G}.

THEOREM 3.1. *For any $\phi \in (S)$, we have*
(i) $\mathcal{G}(\Delta_{\mathrm{G}}\phi) = -\Delta_{\mathrm{G}}(\mathcal{G}\phi)$
(ii) $\mathcal{G}(N\phi) = (\Delta_{\mathrm{G}} + N)\mathcal{G}\phi$.

Proof. (i) Let $\phi \in (\mathcal{S})$ be given by $\phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, f_n\rangle$. Then we have

$$\Delta_{\mathrm{G}}\phi(x) = \sum_{n=0}^{\infty}(n+1)(n+2)\langle : x^{\otimes n} :, \tau\hat{\otimes}_2 f_{n+2}\rangle.$$

Hence by (2.1), we obtain that

$$\begin{aligned}
\mathcal{G}(\Delta_{\mathrm{G}}\phi)(y) &= \sum_{n=0}^{\infty}\langle : y^{\otimes n} :, \sum_{m=0}^{\infty}\frac{(n+2m)!}{n!m!2^m}(-i)^{n+2m}\tau^{\otimes m}\hat{\otimes}_{2m} \\
&\qquad \times \big((n+2m+1)(n+2m+2)\tau\hat{\otimes}_2 f_{n+2m+2}\big)\rangle \\
&= -\sum_{n=0}^{\infty}(n+1)(n+2)\langle : y^{\otimes n} :, \sum_{m=0}^{\infty}\frac{(n+2m+2)!}{(n+2)!m!2^m} \\
&\qquad \times (-i)^{n+2m+2}\tau^{\otimes m+1}\hat{\otimes}_{2(m+1)}\big) f_{n+2m+2}\rangle \\
&= -\Delta_{\mathrm{G}}(\mathcal{G}\phi)(x).
\end{aligned}$$

Thus we have $\mathcal{G}(\Delta_{\mathrm{G}}\phi) = -\Delta_{\mathrm{G}}(\mathcal{G}\phi)$.

(ii) For any exponential vector $\phi_\xi(x) = \sum_{n=0}^\infty \langle : x^{\otimes n} :, \frac{\xi^{\otimes n}}{n!}\rangle$, $\xi \in E$, we have $N\phi_\xi(x) = \sum_{n=0}^\infty n\langle : x^{\otimes n} :, \frac{\xi^{\otimes n}}{n!}\rangle$. Hence we have

$$\begin{aligned}
&\mathcal{G}(N\phi_\xi)(x)\\
&= \sum_{n=0}^\infty \langle : x^{\otimes n} :, \sum_{m=0}^\infty \frac{(n+2m)!}{n!m!2^m}(-i)^{n+2m}\tau^{\otimes m}\hat{\otimes}_{2m}((n+2m)\frac{\xi^{\otimes n+2m}}{(n+2m)!})\rangle\\
&= \sum_{n=0}^\infty \langle : x^{\otimes n} :, \sum_{m=0}^\infty \frac{n}{n!m!2^m}(-i)^{n+2m}\tau^{\otimes m}\hat{\otimes}_{2m}\xi^{\otimes n+2m}\rangle\\
&\qquad + \sum_{n=0}^\infty \langle : x^{\otimes n} :, \sum_{m=0}^\infty \frac{2m}{n!m!2^m}(-i)^{n+2m}\tau^{\otimes m}\hat{\otimes}_{2m}\xi^{\otimes n+2m}\rangle\\
&= \exp\{-\frac{1}{2}|\xi|_0^2\}\sum_{n=0}^\infty n\langle : x^{\otimes n} :, \frac{(-i\xi)^{\otimes n}}{n!}\rangle\\
&\qquad - |\xi|_0^2\exp\{-\frac{1}{2}|\xi|_0^2\}\sum_{n=0}^\infty \langle : x^{\otimes n} :, \frac{(-i\xi)^{\otimes n}}{n!}\rangle.
\end{aligned}$$

By noting that

$$\Delta_{\mathrm{G}}(\mathcal{G}\phi_\xi)(x) = -|\xi|_0^2\exp\{-\frac{1}{2}|\xi|_0^2\}\sum_{n=0}^\infty \langle : x^{\otimes n} :, \frac{(-i\xi)^{\otimes n}}{n!}\rangle$$

and

$$N(\mathcal{G}\phi_\xi)(x) = \exp\{-\frac{1}{2}|\xi|_0^2\}\sum_{n=0}^\infty n\langle : x^{\otimes n} :, \frac{(-i\xi)^{\otimes n}}{n!}\rangle,$$

we have $\mathcal{G}(N\phi_\xi) = (\Delta_{\mathrm{G}} + N)\mathcal{G}\phi_\xi$. But since $\{\phi_\xi; \xi \in \mathcal{S}(\mathbb{R})\}$ spans a dense subspace of $(\mathcal{S})$, it follows from the continuty of Δ_{G}, N and the $\mathcal{G}$-transform that for any $\phi \in (S)$, we have $\mathcal{G}(N\phi) = (\Delta_{\mathrm{G}} + N)\mathcal{G}\phi$.

THEOREM 3.2. *For any Hida-distribution Φ, we have*
(i) $\mathcal{F}(\Delta_{\mathrm{G}}^*\Phi) = -\Delta_{\mathrm{G}}^*\mathcal{F}\Phi$
(ii) $\mathcal{F}(N\Phi) = N\mathcal{F}\Phi + \Delta_{\mathrm{G}}^*\mathcal{F}\Phi$.

Proof. By the duality of Theorem 3.1, for any $\phi \in (\mathcal{S})$ we have

$$\begin{aligned}
\text{(i)}\quad \langle\langle \Delta_{\mathrm{G}}^*\mathcal{F}\Phi, \phi\rangle\rangle &= \langle\langle \Phi, \mathcal{G}(\Delta_{\mathrm{G}}\phi)\rangle\rangle = \langle\langle \Phi, -\Delta_{\mathrm{G}}\mathcal{G}\phi\rangle\rangle = \langle\langle -\Delta_{\mathrm{G}}^*\Phi, \mathcal{G}\phi\rangle\rangle\\
&= \langle\langle -\mathcal{F}(\Delta_{\mathrm{G}}^*)\Phi, \phi\rangle\rangle.
\end{aligned}$$

(ii) $$\begin{aligned}\langle\langle \mathcal{F}(N\Phi),\phi\rangle\rangle &= \langle\langle N\Phi,\mathcal{G}\phi\rangle\rangle = \langle\langle \Phi, N\mathcal{G}\phi\rangle\rangle = \langle\langle \Phi,\mathcal{G}(N\phi)-\Delta_{\mathrm{G}}\mathcal{G}\phi\rangle\rangle \\ &= \langle\langle \Phi,\mathcal{G}(N\phi)+\mathcal{G}(\Delta_{\mathrm{G}}\phi)\rangle\rangle = \langle\langle \mathcal{F}\Phi, N\phi+\Delta_{\mathrm{G}}\phi\rangle\rangle \\ &= \langle\langle N\mathcal{F}\Phi+\Delta_{\mathrm{G}}^{*}\mathcal{F}\Phi,\phi\rangle\rangle.\end{aligned}$$

This completes the proof.

THEOREM 3.3. *For any $\phi \in (\mathcal{S})$, $\sigma_{\mathrm{e}^{-t}}\phi \in (\mathcal{S})$ satisfies the Cauchy problem* (3.1), *where $\sigma_\lambda\phi(x) = \phi(\lambda x)$.*

Proof. Let $v(y,t) = \mathcal{G}u(y,t)$. Then by Theorem 3.1, $v(y,t)$ satisfies the following equation:

$$\frac{\partial}{\partial t}v(y,t) = -Nv(y,t), \qquad v(y,0) = \mathcal{G}\phi(y). \tag{3.3}$$

It is well-known [5] that $q_t(\mathcal{G}\phi)(y) = \int_{S'(\mathbb{R})}\mathcal{G}\phi(\mathrm{e}^{-t}y+\sqrt{1-\mathrm{e}^{-2t}}x)d\mu(x)$ satisfies the equation (3.3). Hence $\mathcal{G}^{-1}(q_t(\mathcal{G}\phi))(x)$ satisfies the equation (3.1). Note that for any $\xi \in E_{\mathbb{C}}$,

$$\mathcal{G}\phi_\xi = \mathrm{e}^{-\frac{1}{2}\langle\xi,\xi\rangle}\phi_{-i\xi}, \qquad q_t(\mathcal{G}\phi_\xi) = \mathrm{e}^{-\frac{1}{2}\langle\xi,\xi\rangle}\phi_{-i\mathrm{e}^{-t}\xi} \tag{3.4}$$

and

$$\sigma_\lambda\phi_\xi(x) = \phi_\xi(\lambda x) = \mathrm{e}^{\frac{1}{2}(\lambda^2-1)\langle\xi,\xi\rangle}\phi_{\lambda\xi}. \tag{3.5}$$

Thus by (2.2), (3.4) and (3.5), we obtain that

$$\begin{aligned}\mathcal{G}^{-1}(q_t(\mathcal{G}\phi_\xi)) &= \mathrm{e}^{\frac{1}{2}(\mathrm{e}^{-2t}-1)\langle\xi,\xi\rangle}\phi_{\mathrm{e}^{-t}\xi} \\ &= \sigma_{\mathrm{e}^{-t}}\phi_\xi, \quad \xi \in E_{\mathbb{C}}.\end{aligned}$$

But since $\{\phi_\xi : \xi \in E_{\mathbb{C}}\}$ spans a dense subspace of $(\mathcal{S})$, it follows from the continuity of σ_λ [16] that for any $\phi \in (\mathcal{S})$, we have $\mathcal{G}^{-1}(q_t(\mathcal{G}\phi)) = \sigma_{\mathrm{e}^{-t}}\phi$. Hence we complete the proof.

REMARK. It is well-known that $\Delta_G + N = \int_{\mathbb{R}}(\partial_t + \partial_t^*)\partial_t dt = \int_{\mathbb{R}} x(t)\partial_t dt$. Thus $\Delta_G + N$ is an infinite dimensional analogue of a first order differential operator $\sum_{j=1}^{n} x_j\frac{\partial}{\partial x_j}$ on $\mathbb{R}^n$. Hence the solution $u(t,x) = \phi(\mathrm{e}^{-t}x)$ of the Cauchy problem (3.1) is indeed the solution of a first order differential equation with variable coefficients in white noise analysis setting.

EXAMPLE. Let B be a bounded operator from $\mathcal{S}'(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. Consider the following Cauchy problem

$$(3.6)\qquad \frac{\partial}{\partial t}u(x,t) = -(\Delta_G + N)u(x,t), \qquad u(x,0) = \langle x, Bx\rangle.$$

Since $(L^2(\mathbb{R}), \mathcal{S}'_p(\mathbb{R}))$ is an abstract Wiener space for any $p > \frac{1}{2}$, B is a trace class operator of $L^2(\mathbb{R})$ and μ has the support contained in $\mathcal{S}'_1(\mathbb{R}) = \mathcal{S}_{-1}(\mathbb{R})$. Hence

$$\mathcal{G}(\langle \cdot, B\cdot\rangle)(x) = trace_{L^2(\mathbb{R})}B - \langle x, Bx\rangle.$$

Therefore, we have

$$\begin{aligned} v(x,t) &= trace_{L^2(\mathbb{R})}B \\ &\quad - \int_{S_{-1}(\mathbb{R})} \langle \mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}y, B(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}y)\rangle d\mu(y) \\ &= trace_{L^2(\mathbb{R})}B - (1-\mathrm{e}^{-2t})trace_{L^2(\mathbb{R})}B - \mathrm{e}^{-2t}\langle x, Bx\rangle \\ &= \mathrm{e}^{-2t}trace_{L^2(\mathbb{R})}B - \mathrm{e}^{-2t}\langle x, Bx\rangle. \end{aligned}$$

Hence $u(x,t) = \mathcal{G}^{-1}(v(\cdot,t)(x)$ satisfies the equation (3.6) and is given by

$$\begin{aligned} u(x,t) &= \mathcal{G}^{-1}(v(\cdot,t))(x) \\ &= \mathrm{e}^{-2t}trace_{L^2(\mathbb{R})}B - \mathrm{e}^{-2t}\int_{S_{-1}(\mathbb{R})} \langle y + ix, B(y+ix)\rangle d\mu(y) \\ &= \mathrm{e}^{-2t}\langle x, Bx\rangle. \end{aligned}$$

THEOREM 3.4. *For any $\Phi \in (\mathcal{S})^*$ with $\Phi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, F_n\rangle$, the Hida distribution*

$$u(x,t) = \sum_{n=0}^{\infty}\langle : x^{\otimes n} :, \sum_{m=0}^{[\frac{n}{2}]}\frac{1}{m!2^m}(\sum_{l=0}^{m}\binom{m}{l}(-1)^l e^{-(n-2l)t})F_{n-2m}\hat{\otimes}\tau^{\otimes l}\rangle$$

satisfies the equation (3.2).

Proof. By taking the Fourier transform in the equation (3.2), we have

$$(3.7)\qquad \frac{\partial}{\partial t}v(y,t) = -Nv(y,t), \qquad v(y,0) = \mathcal{F}\Phi(y),$$

where $v(y,t) = \mathcal{F}(u(\cdot,t))(y)$. Since by (2.3)

$$\mathcal{F}\Phi(y) = \sum_{n=0}^{\infty} \langle : y^{\otimes n} :, \sum_{m=0}^{[\frac{n}{2}]} \frac{(-i)^n}{m!2^m} F_{n-2m} \hat{\otimes} \tau^{\otimes m} \rangle,$$

we can easily check that

$$v(y,t) = \sum_{n=0}^{\infty} \mathrm{e}^{-nt} \langle : y^{\otimes n} :, \sum_{m=0}^{[\frac{n}{2}]} \frac{(-i)^n}{m!2^m} F_{n-2m} \hat{\otimes} \tau^{\otimes m} \rangle$$

satisfies the equation (3.7). By taking the inverse Fourier transform, we have

$$u(x,t) = \mathcal{F}^{-1} v(\cdot,t)(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \sum_{m=0}^{[\frac{n}{2}]} \frac{i^n}{m!2^m} G_{n-2m} \hat{\otimes} \tau^{\otimes m} \rangle,$$

where $G_n = \mathrm{e}^{-nt} \sum_{m=0}^{[\frac{n}{2}]} \frac{(-i)^n}{m!2^m} F_{n-2m} \hat{\otimes} \tau^{\otimes m}$. And we note that

$$\begin{aligned}
&\sum_{m=0}^{[\frac{n}{2}]} \frac{i^n}{m!2^m} G_{n-2m} \hat{\otimes} \tau^{\otimes m} \\
&= \sum_{m=0}^{[\frac{n}{2}]} \frac{i^n}{m!2^m} \Big(\mathrm{e}^{-(n-2m)t} \sum_{l=0}^{[\frac{n-2m}{2}]} \frac{(-i)^{n-2m}}{l!2^l} F_{n-2m-2l} \hat{\otimes} \tau^{\otimes l} \Big) \hat{\otimes} \tau^{\otimes m} \\
&= \sum_{m=0}^{[\frac{n}{2}]} \sum_{l=0}^{[\frac{n-2m}{2}]} \frac{(-1)^m}{l!m!2^{m+l}} \Big(\mathrm{e}^{-(n-2m)t} F_{n-2m-2l} \hat{\otimes} \tau^{\otimes l+m} \Big) \\
&= \sum_{k=0}^{[\frac{n}{2}]} \frac{1}{k!2^k} \Big(\sum_{s=0}^{k} \binom{k}{s} (-1)^s \mathrm{e}^{-(n-2s)t} \Big) F_{n-2k} \hat{\otimes} \tau^{\otimes k}.
\end{aligned}$$

Hence

$$u(x,t) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} :, \sum_{m=0}^{[\frac{n}{2}]} \frac{1}{m!2^m} \Big(\sum_{l=0}^{m} \binom{m}{l} (-1)^l \mathrm{e}^{-(n-2l)t} \Big) F_{n-2m} \hat{\otimes} \tau^{\otimes l} \rangle$$

satisfies the equation (3.2).

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