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THE RELATION BETWEEN THE BERGMAN KERNEL AND THE SZEGŐ KERNEL

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We can expect a close relationship between the Bergman kernel and the Szegő kernel of a domain because we can change boundary integrals to solid integrals via Green's identity. Indeed, the kernels are very closely related. By inspecting the relation of them we can have the information about the Szegő kernel from the information about the Bergman kernel and fruitful applications in mapping problems. For example, by using the relation of the Bergman kernel and the Szegő kernel, we get the transformation formula for the Szegő kernel under certain proper holomorphic mapping from the transformation formula for the Bergman kernel and we use it to characterize proper rational mapping (see Jeong [8]). The purpose of this paper is to reveal a relationship between the Bergman kernel and the Szegő kernel.

Let Ω be a bounded n-connected domain in $\mathbb{C}$ with C^∞ smooth boundary where $n \geq 2$. Let $K(z,w)$ denote the Bergman kernel and $S(z,w)$ denote the Szegő kernel associated with Ω. The following formula by Bergman [6; p.116]

$$K(z,w) = 4\pi S(z,w)^2 + \sum_{i=1}^{n-1} \lambda_i(w) W_i'(z),$$

where W_i' is the derivative of a multi-valued holomorphic function W_i given by analytically continuing around Ω a germ of $\omega_i + \omega_i^*$ where ω_i^* is a local harmonic conjugate for the harmonic measure function ω_i, represents the relation between the Bergman kernel and the Szegő kernel. Bell [4] constructed another formula representing the relationship

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between the above two kernels via Garabedian kernel. It is as follows:

$$K(z,a) = 4\pi S(z,a)^2 + 2\pi \sum_{i=1}^{n-1} \frac{K(a_i,a)}{S'(a_i,a)} L(z,a_i)S(z,a)$$

where $L(z,a)$ denotes the Garabedian kernel defined by Garabedian [7] and for fixed $a \in \Omega$, the points a_i are the zeroes of the function $S(z,a)$.

In this paper, we study a proper anti-holomorphic correspondence and make a formula relating the Bergman kernel to the Szegő kernel in a different form. Before we state and prove our main theorem, we must explain some notations and basic facts.

For a bounded domain Ω in $\mathbb{C}$ with C^∞ smooth boundary, let Ω^* be the set of points $\bar{w}$ such that $w \in \Omega$. The Szegő kernel $S(z,w)$ associated to Ω defines an analytic subvariety $V := \{(z,\bar{w}) \in \Omega \times \Omega^* : S(z,w) = 0\}$. The projection maps $\pi_1 : V \to \Omega$ and $\pi_2 : V \to \Omega^*$ are proper holomorphic (see Bell [4] and Bell [5; p. 106]). Hence, V is identified with $\overline{f}$ where the multi-valued map $f = \overline{\pi}_2 \circ \pi_1^{-1}$ is a proper anti-holomorphic self-correspondence of Ω.

Let Ω be a bounded domain in $\mathbb{C}^n$. Let $H(\Omega)$ denote the space of holomorphic functions in $L^2(\Omega)$ and $AH(\Omega) = \{\varphi \in L^2(\Omega) : \overline{\varphi} \in H(\Omega)\}$. Since $AH(\Omega)$ is a closed subspace of $L^2(\Omega)$, there exists a unique orthogonal projection Q of $L^2(\Omega)$ onto $AH(\Omega)$ and it satisfies $Q\varphi = \overline{P\overline{\varphi}}$ for $\varphi \in L^2(\Omega)$ where P is the Bergman projection of $L^2(\Omega)$ onto $H(\Omega)$. It is easy to see that $Q\varphi(z) = \int_\Omega \overline{K}(z,w)\varphi(w)\, dV_w$ for every $\varphi \in L^2(\Omega)$ where $K(z,w)$ denotes the Bergman kernel associated with Ω.

The transformation formula for the Bergman kernel under proper holomorphic correspondence is already known (see Bedford and Bell [1]). Now, by a little modification of it, we will make a transformation formula for the Bergman kernel under proper anti-holomorphic correspondence. First we suppose that Ω_1 and Ω_2 are bounded domains in $\mathbb{C}^n$ and that f is a bi-anti-holomorphic mapping of Ω_1 onto Ω_2. Let $u(z) = det[\frac{\partial f}{\partial \bar{z}}]$. For each $\varphi \in L^2(\Omega_2)$, define $\Lambda\varphi(z) = u(z) \cdot (\overline{\varphi} \circ f)(z)$. We see that Λ is an isometric anti-isomorphism of $L^2(\Omega_2)$ onto $L^2(\Omega_1)$, i.e., it is isometric one-to-one, conjugate-linear map of $L^2(\Omega_2)$ onto $L^2(\Omega_1)$. It preserves anti-holomorphic functions. Consequently, we get the following proposition.

PROPOSITION 1. *Let Ω_1 and Ω_2 be bounded domains in $\mathbb{C}^n$ and $f : \Omega_1 \to \Omega_2$ be a bi-anti-holomorphic mapping. Let $K_i(z,w)$ denote the Bergman kernel associated to Ω_i for $i = 1, 2$. The Bergman kernels transform according to*

$$K_1(z,w) = u(w)K_2(f(w), f(z))\overline{u}(z)$$

for every $z, w \in \Omega_1$.

Proof. Let $\{v_j\}_{j=1}^{\infty}$ denote an orthogonal basis for $AH(\Omega_2)$. Then $\{u\cdot(\overline{v}_j \circ f)\}_{j=1}^{\infty}$ is an orthogonal basis for $AH(\Omega_1)$. Note that $\overline{K}_1(\cdot, z) \in AH(\Omega_1)$ when $z \in \Omega_1$. It follows that

$$\begin{aligned}\overline{K}_1(w,z) &= \sum_{j=1}^{\infty} u(w)(\overline{v}_j \circ f)(w)\overline{u}(z)(v_j \circ f)(z) \\ &= u(w)\Big\{\sum_{j=1}^{\infty} \overline{v}_j(f(w))v_j(f(z))\Big\}\overline{u}(z) \\ &= u(w)\overline{K}_2(f(z), f(w))\overline{u}(z).\end{aligned}$$

Therefore, by the conjugate-symmetry properties of the Bergman kernels,

$$K_1(z,w) = u(w)K_2(f(w), f(z))\overline{u}(z)$$

for z, $w \in \Omega_1$. □

We want to move our concern to the case when $f : \Omega_1 \multimap \Omega_2$ is a proper anti-holomorphic correspondence between two bounded domains in $\mathbb{C}^n$. Let Ω_2^* denote the set of points w such that $\bar{w} \in \Omega_2$. Then f is given by $f(z) = \{\bar{w} \in \Omega_2 : (z,w) \in V \subset \Omega_1 \times \Omega_2^*\}$ where V is an analytic subvariety of $\Omega_1 \times \Omega_2^*$. The projections $\pi_1 : V \to \Omega_1$ and $\pi_2 : V \to \Omega_2^*$ are proper maps. There are subvarieties V_1 and V_2 of Ω_1 and Ω_2^*, respectively and positive integers p and q satisfying the following conditions:
(1) Near a point $z \in \Omega_1 - V_1$, there are exactly p anti-holomorphic mappings $\{f_i\}_{i=1}^{p}$ defined near z that represent the multi-valued mapping $\overline{\pi}_2 \circ \pi_1^{-1}$
(2) Near a point $w \in \Omega_2 - V_2^*$ where $V_2^* = \{\zeta \in \Omega_2 : \bar{\zeta} \in V_2\}$, there are q anti-holomorphic mappings $\{F_j\}_{j=1}^{q}$ that represent $\pi_1 \circ \overline{\pi}_2^{\,-1}$.

Let $u_i(z) = det[\frac{\partial f_i}{\partial \bar{z}}]$ and $U_j(w) = det[\frac{\partial F_j}{\partial \bar{w}}]$. Let Q_i denote the orthogonal projection of $L^2(\Omega_i)$ onto $AH(\Omega_i)$, $i = 1, 2$. By modifying Theorem 2 in Bedford and Bell [2], we can obtain the following lemma.

LEMMA 2. *Suppose $f : \Omega_1 \multimap \Omega_2$ is a proper anti-holomorphic correspondence between two bounded domains in $\mathbb{C}^n$. If $\varphi \in L^2(\Omega_2)$, then*

$$\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i) \in L^2(\Omega_1)$$

$$\text{and} \qquad Q_1\Big(\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i)\Big) = \sum_{i=1}^{p} u_i(\overline{Q}_2\varphi \circ f_i).$$

Proof. We already know that $\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i)$ is well defined on $\Omega - V_1$. So we get

$$\Big\|\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i)\Big\|^2_{L^2(\Omega_1 - V_1)} \leq p \int_{\Omega_1 - V_1} \sum_{i=1}^{p} |u_i|^2 |\overline{\varphi} \circ f_i|^2$$

$$= pq \int_{\Omega_2 - V_2^*} |\overline{\varphi}|^2 = pq\|\varphi\|^2_{L^2(\Omega_2)}.$$

Since V_1 is a set of measure 0 in Ω_1, we see that $\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i)$ extends to be in $L^2(\Omega_1)$ and that

$$\Big\|\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i)\Big\|_{L^2(\Omega_1)} \leq \sqrt{pq}\|\varphi\|_{L^2(\Omega_2)}.$$

If $\varphi \in AH(\Omega_2)$, then $\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i) \in AH(\Omega_1 - V_1) \cap L^2(\Omega_1)$ implies that $\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i) \in AH(\Omega_1)$ by the L^2 Removable Singularity Theorem (see Bell [3]). So $Q_1(\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i)) = \sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i) = \sum_{i=1}^{p} u_i(\overline{Q}_2\varphi \circ f_i)$.

Therefore, we will be finished if we show that the transformation formula holds for $\varphi \in L^2(\Omega_2)$ that is orthogonal to $AH(\Omega_2)$. If φ is orthogonal to $AH(\Omega_2)$ and $g \in AH(\Omega_1)$, then $\sum_{j=1}^{q} U_j(\overline{g} \circ F_j) \in AH(\Omega_2)$ implies that

$$\int_{\Omega_1} \left(\sum_{i=1}^{p} u_i(\overline{\varphi} \circ f_i)\right) \overline{g} = \int_{\Omega_2} \overline{\varphi} \left(\sum_{j=1}^{q} U_j(\overline{g} \circ F_j)\right) = 0.$$

So $\sum_{i=1}^{p} u_i(\overline{\varphi}\circ f_i)$ is orthogonal to $AH(\Omega_1)$ and hence $Q_1(\sum_{i=1}^{p} u_i(\overline{\varphi}\circ f_i)) = 0$. This completes the proof of Lemma 2. □

By using the above lemma, we get the following transformation formula for the Bergman kernel under the proper anti-holomorphic correspondence.

PROPOSITION 3. *Assume the same hypothesis as in Lemma 2. Let $K_i(z,w)$ denote the Bergman kernel associated to Ω_i for $i=1,2$. The Bergman kernels transform according to*

$$\sum_{i=1}^{p} u_i(z)K_2(f_i(z),w) = \sum_{j=1}^{q} U_j(w)K_1(F_j(w),z)$$

for all $z\in\Omega_1$ and $w\in\Omega_2$.

Proof. For $w\in\Omega_2 - V_2^*$, there exist an arbitrarily small neighborhood W of w in $\Omega_2 - V_2^*$ and $F_j : W\to D_j$ which are bi-anti-holomorphic mappings for $1\le j\le q$ such that $F_j\circ f_{i(j)} = id_{D_j}$ for some $1\le i(j)\le p$ where $\{D_j\}_{j=1}^{q}\subset\Omega_1 - V_1$ are disjoint and for each $1\le j\le q$, $\{f_i(D_j)\}_{i=1}^{p}$ are disjoint. Take $\theta_w\in C_0^\infty(W)$ which is radially symmetric about w with $\int_{\Omega_2}\theta_w = 1$. For any $h\in AH(\Omega_2 - V_2^*)$, we have $h(w) = \int_{\Omega_2} h\theta_w$. As a result, we get $Q_2\theta_w(\cdot) = \overline{K}_2(\cdot,w)$. Now, for $z\in\Omega_1 - V_1$,

$$\sum_{i=1}^{p} u_i(z)K_2(f_i(z),w) = \sum_{i=1}^{p} u_i(z)(\overline{Q}_2\theta_w\circ f_i)(z)$$

$$=Q_1(\sum_{i=1}^{p} u_i(\overline{\theta}_w\circ f_i))(z) \qquad \text{by Lemma 2}$$

$$=\int_{\Omega_1}\overline{K}_1(z,\xi)\sum_{i=1}^{p} u_i(\xi)(\overline{\theta}_w\circ f_i)(\xi)\,dV_\xi$$

$$=\sum_{j=1}^{q}\int_{D_j}\overline{K}_1(z,F_j\circ f_{i(j)}(\xi))\sum_{i=1}^{p} u_i(\xi)(\theta_w\circ f_i)(\xi)\,dV_\xi$$

$$=\sum_{j=1}^{q}\int_{D_j} K_1(F_j\circ f_{i(j)}(\xi),z)|u_{i(j)}(\xi)|^2 U_j(f_{i(j)}(\xi))(\theta_w\circ f_{i(j)})(\xi)\,dV_\xi$$

$$= \sum_{j=1}^{q} \int_{W} K_1(F_j(\eta), z) U_j(\eta) \theta_w(\eta)\, dV_\eta$$

$$= \sum_{j=1}^{q} \int_{\Omega_2} K_1(F_j(\eta), z) U_j(\eta) \theta_w(\eta)\, dV_\eta = \sum_{j=1}^{q} K_1(F_j(w), z) U_j(w)$$

since $K_1(F_j(\cdot), z)U_j(\cdot) \in AH(\Omega_2 - V_2^*)$. Hence,

$$\sum_{i=1}^{p} u_i(z) K_2(f_i(z), w) = \sum_{j=1}^{q} U_j(w) K_1(F_j(w), z)$$

for $w \in \Omega_2 - V_2^*$ and $z \in \Omega_1 - V_1$. By the L^2 Removable Singularity theorem, it holds for all $w \in \Omega_2$ and $z \in \Omega_1$. □

Now we are ready to prove the following main theorem relating the Bergman kernel to the Szegő kernel.

THEOREM 4. *Suppose that Ω is a bounded n-connected domain in $\mathbb{C}$ with C^∞ smooth boundary where $n \geq 2$. The Bergman kernel $K(z, w)$ associated to Ω is related to the Szegő kernel via the identity*

$$\sum_{i=1}^{n-1} K(a_i, w) \frac{S_{\bar{a}}(a_i, a)}{S_z(a_i, a)} = \sum_{j=1}^{n-1} K(w_j, a) \frac{S_{\bar{w}}(w_j, w)}{S_z(w_j, w)}$$

where for fixed $a \in \Omega$, $w \in \Omega$, the points $\{a_i\}_{i=1}^{n-1}$ and $\{w_j\}_{j=1}^{n-1}$ are the zeroes of the functions $S(z, a)$ and $S(z, w)$, respectively. Note that $S_z(a_i, a) = \frac{\partial}{\partial z} S(z, a)|_{z=a_i}$ and $S_z(w_j, w) = \frac{\partial}{\partial z} S(z, w)|_{z=w_j}$.

Proof. For $a \in \Omega$, $S(z, a)$ has $n-1$ zeroes $\{a_i\}_{i=1}^{n-1}$ in Ω with multiplicity. The multi-valued map $a \mapsto a_1, \ldots, a_{n-1}$ is a proper anti-holomorphic self-correspondence of Ω. Let f denote this multi-valued map. There exists a subvariety V_1 of Ω such that $\{f_i\}_{i=1}^{n-1}$ denote the mappings that locally define f and $f_i(a) = a_i$ for $a \in \Omega - V_1$. It is obvious that the inverse correspondence f^{-1} is equal to f.

For $a \in \Omega - V_1$, let γ_{a_i} in Ω be a sufficiently small simple closed curve surrounding a_i not to include other zeroes of $S(z, a)$ inside or on γ_{a_i}. By using the fact that

$$z \frac{S_z(z, a)}{S(z, a)} = \sum_{j=1}^{n-1} \frac{z}{z - a_j} + z \frac{F'(z)}{F(z)}$$

where $F(z)$ is holomorphic with no zeroes inside or on γ_{a_i}, we can represent f_i by

$$f_i(a) = \frac{1}{2\pi i}\int_{\gamma_{a_i}} z\frac{S_z(z,a)}{S(z,a)}\,dz.$$

Hence

$$\begin{aligned}
\frac{\partial f_i}{\partial \bar{a}}(a) &= \frac{1}{2\pi i}\int_{\gamma_{a_i}} z\frac{\partial}{\partial \bar{a}}\left(\frac{S_z(z,a)}{S(z,a)}\right) dz \\
&= \frac{1}{2\pi i}\int_{\gamma_{a_i}} z\left(\frac{S_{\bar{a}z}(z,a)}{S(z,a)} - \frac{S_z(z,a)S_{\bar{a}}(z,a)}{S(z,a)^2}\right) dz \\
&= \frac{1}{2\pi i}\int_{\gamma_{a_i}} z\frac{\partial}{\partial z}\left(\frac{S_{\bar{a}}(z,a)}{S(z,a)}\right) dz \\
&= \frac{1}{2\pi i}\int_{\gamma_{a_i}} \frac{\partial}{\partial z}\left(z\frac{S_{\bar{a}}(z,a)}{S(z,a)}\right) dz - \frac{1}{2\pi i}\int_{\gamma_{a_i}} \frac{S_{\bar{a}}(z,a)}{S(z,a)}\,dz \\
&= -\frac{1}{2\pi i}\int_{\gamma_{a_i}} \frac{S_{\bar{a}}(z,a)}{S(z,a)}\,dz \\
&= -\frac{S_{\bar{a}}(a_i,a)}{S_z(a_i,a)} \qquad \text{by the Residue Theorem.}
\end{aligned}$$

Since f is a proper anti-holomorphic correspondence and $\{f_i\}_{i=1}^{n-1}$ define f locally, it follows that by Proposition 3,

$$\sum_{i=1}^{n-1} K(f_i(a),w)\frac{\partial f_i}{\partial \bar{a}}(a) = \sum_{j=1}^{n-1} K(f_j(w),a)\frac{\partial f_j}{\partial \bar{w}}(w).$$

Therefore,

$$\sum_{i=1}^{n-1} K(a_i,w)\frac{S_{\bar{a}}(a_i,a)}{S_z(a_i,a)} = \sum_{j=1}^{n-1} K(w_j,a)\frac{S_{\bar{w}}(w_j,w)}{S_z(w_j,w)}$$

for $a, w \in \Omega - V_1$. By the L^2 Removable Singularity theorem, it holds for all $a, w \in \Omega$. $\square$

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