A CHANGE OF SCALE FORMULA
FOR WIENER INTEGRALS ON THE
PRODUCT ABSTRACT WIENER SPACES

YOUNG SIK KIM, JAE MOON AHN, KUN SOO CHANG AND IL YOO

1. Introduction

It has long been known that Wiener measure and Wiener measurbility behave badly under the change of scale transformation [3] and under translation [2]. However, Cameron and Storvick [4] obtained the fact that the analytic Feynman integral was expressed as a limit of Wiener integrals for a rather larger class of functionals on a classical Wiener space. And then they found a rather nice change of scale formula for Wiener integrals on a classical Wiener space [5]. In [10,11,12], Yoo, Yoon and Skoug extended these results to Yeh-Wiener space and to an abstract Wiener space.

The purpose of this paper is to show the existence of the analytic Feynman integral for certain functionals on the product abstract Wiener space and to obtain the relationships between the analytic Feynman integrals and the abstract Wiener integrals. And then using these results, we obtain change of scale formulas for abstract Wiener integrals on the product abstract Wiener space. Finally, we will show that most of theorems given in [10] become corollaries of our theorems.

2. Definitions and Preliminaries

Let H be a real separable infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$. Let $\|\cdot\|_0$ be a measurable norm on H with respect to the Gaussian cylinder set measure μ on H.

Let B denote the completion of H with respect to $\|\cdot\|_0$. Then it is well known [9] that B becomes a separable Banach space. Let i denote the natural injection from H into B. The adjoint operator i^* is one-to-one and maps B^* continuously onto a dense subset of H^*. By identifying H^* with H and B^* with i^*B^*, we have a triplet (B^*, H, B) in such a way that $B^* \subset H^* \equiv H \subset B$ and $\langle y, x \rangle = (y, x)$ for all $y \in B^*$ and $x \in H$, where $(\cdot, \cdot)$ denotes the natural dual pairing between B^* and B. By a well-known result of Gross, $\mu \circ i^{-1}$ has a unique countably additive extension m to the Borel σ-algebra $\mathcal{B}(B)$ on B. Then we say that the triplet (B, H, m) is an abstract Wiener space and m is an abstract Wiener measure. For more details, see [9].

Let F be a complex-valued measurable functional on (B, H, m). Then we will denote the abstract Wiener integral of F with respect to m by

$$\int_B F(x)\, dm(x).$$

Let $\langle e_n \rangle$ denote a complete orthonormal system on H such that $e_n's$ are in B^*. For each $h \in H$ and $x \in B$, define a stochastic inner product $(\cdot, \cdot)^\sim$ between H and B as follows :

$$(2.1) \qquad (h, x)^\sim = \begin{cases} \lim_{n \to \infty} \sum_{k=1}^n \langle h, e_k \rangle (e_k, x), & \text{if the limit exists} \\ 0, & \text{otherwise} \end{cases}$$

It is well known [8] that for every $h \in H$, $(h, x)^\sim$ exists for $m - a.e$ $x \in B$, and is a Borel measurable function on B having a Gaussian distribution with mean zero and variance $|h|^2$. Also if both h and x are in H, then $(h, x)^\sim = \langle h, x \rangle$. Furthermore, it is easy to show that for each $\alpha \in \mathbb{R}, (\alpha h, x)^\sim = \alpha(h, x)^\sim = (h, \alpha x)^\sim$, $x \in B$, hold and that $(h, x)^\sim = (h, x)$ m-a.e. $x \in B$ if $h \in B^*$. It is well known [8] that $(h, x)^\sim$ is essentially independent of the choice of the complete orthonormal system used in its definition.

DEFINITION 2.1. Let $B^\nu = \times_{j=1}^\nu B$ denote the product of ν copies of B and let F be a complex-valued measurable functional on B^ν such that the integral

$$(2.2) \qquad J(F; \vec{\lambda}) = \int_{B^\nu} F(\lambda_1^{-\frac{1}{2}} x_1, \cdots, \lambda_\nu^{-\frac{1}{2}} x_\nu)\, dm^\nu(\vec{x})$$

exists for all $\lambda_k > 0$, $k = 1, 2, \cdots, \nu$, where m^ν is the product abstract Wiener measure on B^ν, $\vec{\lambda} = (\lambda_1, \cdots, \lambda_\nu) \in \mathbb{R}^\nu$ and $\vec{x} = (x_1, \cdots, x_\nu) \in B^\nu$. If there exists an analytic function $J^*(F; \vec{z})$ on
on $\Omega = \{\vec{z} = (z_1, \cdots, z_\nu) \in \mathbb{C}^\nu : Re(z_k) > 0 \text{ for } k = 1, 2, \cdots \nu\}$ such that $J^*(F; \vec{\lambda}) = J(F; \vec{\lambda})$ for all $\lambda_k > 0$, $k = 1, 2, \cdots, \nu$, then we define $J^*(F; \vec{z})$ to be the analytic Wiener integral of F over B^ν with parameter $\vec{z}$, and for $\vec{z} \in \Omega$ we write

$$(2.3) \qquad\qquad I^{aw}(F; \vec{z}) = J^*(F; \vec{z}).$$

Let $\vec{q} = (q_1, \cdots, q_\nu) \in \mathbb{R}^\nu$ be such that $q_k \neq 0$ for $k = 1, 2, \cdots, \nu$. If the following limit (2.4) exists, we call it the analytic Feynman integral of F over B^ν with parameter $\vec{q}$ and we write

$$(2.4) \qquad\qquad I^{af}(F; \vec{q}) = \lim_{\vec{z} \to -i\vec{q}} I^{aw}(F; \vec{z}).$$

where $\vec{z} = (z_1, \cdots, z_\nu)$ approaches $-i\vec{q} = (-iq_1, \cdots, -iq_\nu)$ through Ω.

Given two complex-valued functionals F and G on B^ν, we say that $F = G$ s-a.e. if for all $\alpha_k > 0$, $k = 1, \cdots, \nu$, $F(\alpha_1 x_1, \cdots, \alpha_\nu x_\nu) = G(\alpha_1 x_1, \cdots, \alpha_\nu x_\nu)$ for m^ν-a.e. $\vec{x}$ in B^ν. For a functional F on B^ν, we will denote by $[F]$ the equivalence class of functionals which are equal to F s-a.e.

DEFINITION 2.2. Let $\mathcal{F}(B^\nu)$ be a class of all equivalence classes of functionals on B^ν which have the form

$$(2.5) \quad F(\vec{x}) = \int_H \exp\left\{ i \sum_{j=1}^\nu (h, x_j)^\sim \right\} d\mu(h), \ \vec{x} = (x_1, \cdots, x_\nu) \in B^\nu,$$

for some finite complex Borel measure μ on H. In particular, when $\nu = 1$, $\mathcal{F}(B^\nu)$ is reduced to $\mathcal{F}(B)$, which is the Fresnel class on the abstract Wiener apace (B, H, m).

Let $M(H)$ denote the space of all finite complex Borel measures on H. Then it is well known that $M(H)$ is a Banach algebra under convolution, with the norm $\|\mu\|$ equal to the total variation of $\mu \in M(H)$. By using Propositions 3.2 or 5.1 in [8], we can prove that the mapping $\mu \longmapsto [F]$ is a Banach algebra isomorphism where μ and F

are related by (2.5) and hence $\mathcal{F}(B^\nu)$ becomes a Banach algebra under the norm $\|F\| = \|\mu\|$.

The following theorem, which is quoted from [8], is necessary for proving the existence of the analytic Feynman integral for the functionlas F belonging to $\mathcal{F}(B^\nu)$.

THEOREM 2.1. *Let D be an open subset of $\mathbb{C}^k$, where $\mathbb{C}^k = \times_{j=1}^{k} \mathbb{C}$ is the product of k copies of the complex plane $\mathbb{C}$. Assume that $g : D \to \mathbb{C}$ is continuous and analytic in each variable separately. That is, for each j, $1 \leq j \leq k$, and each point $(z_1, \cdots, z_{j-1}, z_{j+1} \cdots, z_k) \in \mathbb{C}^{k-1}$ such that $D_j = \{z_j \in \mathbb{C} : (z_1, \cdots, z_j, \cdots, z_k) \in D\}$ is nonempty, the function $f(z_j) = g(z_1, \cdots, z_j, \cdots, z_k)$ is analytic in D_j. Then g is analytic as a function of k complex variables in D. If D is connected and contains the set $D^+ = \{(z_1, \cdots, z_k) \in \mathbb{C}^k : Re(z_j) > 0, \ 1 \leq j \leq k\}$, then g is uniquely determined by its restriction to D^+.*

3. Change of Scale Formulas

We begin this section with obtaining the existence theorem of the analytic Wiener and Feynman integrals for the functionals F given by (2.5).

Throughout this section, we will let

$$\wedge = \{\vec{\lambda} = (\lambda_1, \cdots, \lambda_\nu) \in \mathbb{R}^\nu : \lambda_j > 0, \ 1 \leq j \leq \nu\},$$

and

$$\Omega = \{\vec{z} = (z_1, \cdots, z_\nu) \in \mathbb{C}^\nu : Re(z_j) > 0, \ 1 \leq j \leq \nu\}.$$

THEOREM 3.1. *Let $F \in \mathcal{F}(B^\nu)$ be given by (2.5). Then the analytic Wiener integral of F over B^ν with parameter $\vec{z} = (z_1, \cdots, z_\nu) \in \Omega$ exists, and*

$$(3.1) \qquad I^{aw}(F; \vec{z}) = \int_H \exp\left\{-\frac{1}{2} \sum_{j=1}^{\nu} \frac{1}{z_j} |h|^2\right\} d\mu(h).$$

Also the analytic Feynman integral of F over B^ν with parameter $\vec{q} = (q_1, \cdots, q_\nu)$ exists, provided that $q_k \neq 0$ for all $k = 1, 2, \cdots, \nu$, and

$$(3.2) \qquad I^{af}(F; \vec{q}) = \int_H \exp\left\{-\frac{i}{2} \sum_{j=1}^{\nu} \frac{1}{q_j} |h|^2\right\} d\mu(h).$$

Proof. Using Fubini's theorem, we have

$$
J(F; \vec{\lambda}) = \int_{B^\nu} \int_H \exp\left\{ i \sum_{j=1}^\nu (h, \lambda_j^{-\frac{1}{2}} x_j)^\sim \right\} d\mu(h) dm^\nu(\vec{x})
$$

$$
= \int_H \int_{B^\nu} \exp\left\{ i \sum_{j=1}^\nu (h, \lambda_j^{-\frac{1}{2}} x_j)^\sim \right\} dm^\nu(\vec{x}) \, d\mu(h)
$$

$$
= \int_H \exp\left\{ -\frac{1}{2} \sum_{j=1}^\nu \frac{1}{\lambda_j} |h|^2 \right\} d\mu(h)
$$

for all $\vec{\lambda} = (\lambda_1, \cdots, \lambda_\nu) \in \wedge$, where the last equality is established from the fact that each $(h, x_j)^\sim$ is Gaussian random variable with mean zero and variance $|h|^2$ with respect to the abstract Wiener measure m. Define

$$
J^*(F; \vec{z}) = \int_H \exp\left\{ -\frac{1}{2} \sum_{j=1}^\nu \frac{1}{z_j} |h|^2 \right\} d\mu(h)
$$

for all $\vec{z} \in \Omega_0 = \left\{ \vec{z} = (z_1, \cdots, z_\nu) \in \mathbb{C}^\nu : z_j \neq 0, \, Re(z_j) \geq 0, \, 1 \leq j \leq \nu \right\}$. Then $J^*(F; \cdot)$ exists for all $\vec{z} \in \Omega_0$, and is continuous on Ω_0 by the dominated convergence theorem. Since $J^*(F; \vec{\lambda}) = J(F; \vec{\lambda})$ for all $\vec{\lambda} \in \wedge$, it is enough to show that the restriction of $J^*(F; \cdot)$ to Ω is an analytic function in order to show that equations (3.1) and (3.2) are established. Using Morera's theorem, we can show that $J^*(F; \cdot)$ is analytic in each complex variable separately in Ω. With the assistance of Theorem 2.1, we can conclude that the restriction of $J^*(F; \cdot)$ to Ω is an analytic function.

The following lemma plays a key role for obtaining relationships between the abstract Wiener integral and the analytic Feynmam integral on an abstract Wiener space (B, H, m), which is quoted from [10].

LEMMA 3.2. Let $z \in \mathbb{C}$ with $Re(z) > 0$, let $\{e_j : j = 1, \cdots, n\}$ be an orthonormal set in H, and let $h \in H$. Then

$$
(3.3) \quad \int_B \exp\left\{ \frac{1-z}{2} \sum_{j=1}^n \left[(e_j, x)^\sim \right]^2 + i(h, x)^\sim \right\} dm(x)
$$

$$
= z^{-\frac{n}{2}} \exp\left\{ \frac{z-1}{2z} \sum_{j=1}^n \left[\langle e_j, h \rangle \right]^2 - \frac{1}{2} |h|^2 \right\}.
$$

In the following two theorems 3.3 and 3.5, for every $F \in \mathcal{F}(B^\nu)$, we express the analytic Wiener integral and the analytic Feynman integral of F over B^ν as a limit of a sequence of abstract Wiener integrals, respectively.

THEOREM 3.3. Let $\langle e_n \rangle$ be a complete orthonormal system in H and let $\vec{z} = (z_1, \cdots, z_\nu) \in \Omega$. Let $F \in \mathcal{F}(B^\nu)$ be given by (2.5). Then we have

(3.4)

$$
I^{aw}(F; \vec{z}) = \lim_{n \to \infty} \left(\prod_{j=1}^{\nu} z_j \right)^{\frac{n}{2}} \int_{B^\cdot} \exp\left\{ \sum_{j=1}^{\nu} \frac{1 - z_j}{2} \sum_{k=1}^{n} [(e_k, x_j)^\sim]^2 \right\}
$$
$$
\cdot F(\vec{x})\, dm^\nu(\vec{x}),
$$

where $dm^\nu(\vec{x}) = dm(x_1)\, dm(x_2) \cdots dm(x_\nu)$, and $\prod_{j=1}^{\nu} z_j = z_1 \cdot z_2 \cdots z_\nu$.

Proof. Since $F \in \mathcal{F}(B^\nu)$ is given by (2.5),

$$
F(\vec{x}) = \int_H \exp\left\{ i \sum_{j=1}^{\nu} (h, x_j)^\sim \right\} d\mu(h), \ \vec{x} \in B^\nu,
$$

for some $\mu \in M(H)$. By Fubini's theorem and Lemma 3.2, we obtain

$$
\int_{B^\nu} \exp\left\{ \sum_{j=1}^{\nu} \frac{1 - z_j}{2} \sum_{k=1}^{n} [(e_k, x_j)^\sim]^2 \right\} F(\vec{x})\, dm^\nu(\vec{x})
$$
$$
= \left(\prod_{j=1}^{\nu} z_j \right)^{-\frac{n}{2}} \int_H \exp\left\{ \sum_{j=1}^{\nu} \frac{z_j - 1}{2 z_j} \sum_{k=1}^{n} [\langle e_k, h \rangle]^2 \right\} \exp\left\{ -\frac{\nu}{2} |h|^2 \right\} d\mu(h).
$$

Next, using the bounded convergence theorem, the equation (3.1) and Parseval's relation, it follows that

$$
\lim_{n \to \infty} \left(\prod_{j=1}^{\nu} z_j \right)^{\frac{n}{2}} \int_{B^\nu} \exp\left\{ \sum_{j=1}^{\nu} \frac{1 - z_j}{2} \sum_{k=1}^{n} [(e_k, x_j)^\sim]^2 \right\} F(\vec{x})\, dm^\nu(\vec{x})
$$
$$
= \int_H \exp\left\{ -\frac{1}{2} \sum_{j=1}^{\nu} \frac{1}{z_j} |h|^2 \right\} d\mu(h)
$$
$$
= I^{aw}(F; \vec{z}).
$$

COROLLARY 3.4. Let $\langle e_n \rangle$ be a complete orthonormal system in H and let $F \in \mathcal{F}(B)$ be given by

$$F(x) = \int_H \exp\{i(h, x)^\sim\} \, d\mu(h), \; x \in B$$

for some $\mu \in M(H)$. Let $z \in \mathbb{C}$ with $Re(z) > 0$. Then the analytic Wiener integral of F over B with parameter z exists, and
(3.5)

$$I^{aw}(F; z) = \lim_{n \to \infty} z^{\frac{n}{2}} \int_B \exp\left\{ \frac{1-z}{2} \sum_{k=1}^n \left[(e_k, x)^\sim \right]^2 \right\} F(x) \, dm(x).$$

Proof. Apply Theorem 3.3 after making the following choices: $\nu = 1, z_1 = z$.

REMARK 1. The first part of Cororally 3.4 coincides with that of Proposition2.2 in [8], and the equation(3.5) of Cororally 3.4 coincides with the equation(3.3) of Theorem 3 in [10] by taking $\langle \lambda_j \rangle_{j=1}^\infty = \langle z \rangle$ therein.

THEOREM 3.5. *Let $\langle e_n \rangle$ be a complete orthonormal system in H and let $F \in \mathcal{F}(B^\nu)$ be given by (2.5). Let $\langle z_{k,n} \rangle$ be a sequence of complex numbers such that $Re(z_{k,n}) > 0$ and $z_{k,n} \to -iq_k(q_k \neq 0)$ as $n \to \infty$ for $k = 1, 2, \cdots, \nu$. Then*

(3.6)

$$I^{af}(F; \vec{q})$$

$$= \lim_{n \to \infty} \left[\prod_{j=1}^\nu z_{j,n} \right]^{\frac{n}{2}} \int_{B^\nu} \exp\left\{ \sum_{j=1}^\nu \frac{1 - z_{j,n}}{2} \sum_{k=1}^n \left[(e_k, x_j)^\sim \right]^2 \right\}$$

$$\cdot F(\vec{x}) \, dm^\nu(\vec{x}),$$

where

$$\vec{q} = (q_1, \cdots, q_\nu), q_k \neq 0 \text{ for } k = 1, 2, \cdots, \nu,$$

$$\prod_{j=1}^\nu z_{j,n} = z_{1,n} z_{2,n} \cdots z_{\nu,n}, \text{ and}$$

$$dm^\nu(\vec{x}) = dm(x_1) dm(x_2) \cdots dm(x_\nu).$$

Proof. To prove this theorem, we modify the proof of Theorem 3.3 by replacing "z_j" by "$z_{j,n}$" wherever it occurs, and by replacing "$-2z_j$" by "$2q_j i$" in the last equality in the proof of Theorem 3.3.

COROLLARY 3.6. *Let $\langle e_n \rangle$ be a complete orthonormal system in H and let $F \in \mathcal{F}(B)$ be given by*

$$F(x) = \int_H \exp\{i(h,x)^\sim\}\, d\mu(h), \quad x \in B,$$

for some $\mu \in \mathcal{M}(H)$. Let $\langle z_n \rangle$ be a sequence of complex numbers such that $Re(z_n) > 0$ for all $n \in \mathbb{N}$, and $z_n \to -iq(q \neq 0)$ as $n \to \infty$. Then the analytic Feynman integral of F over B with parameter q exists, and
(3.7)

$$I^{af}(F;q) = \lim_{n\to\infty} z_n^{\frac{n}{2}} \int_B \exp\left\{\frac{1-z_n}{2}\sum_{k=1}^n \left[(e_k,x)^\sim\right]^2\right\} F(x)\, dm(x).$$

Proof. Apply Theorem 3.5 after making the following choices ; $\nu = 1$, $z_{1,n} = z_n$, and $q_1 = q$.

REMARK 2. The first part of Corollary 3.6 coincides with the second part of Proposition 2.2 in [8], and the equation(3.7) of Corollary 3.6 coincides with the equation(3.2) of Theorem 2 in [10].

Now, we can obtain our main result, namely a change of scale formula for abstract Wiener integrals on a product abstract Wiener space by using Theorem 3.3.

THEOREM 3.7. *Let $\rho_k > 0$, $k = 1, 2, \cdots, \nu$ be given and let $\langle e_n \rangle$ be a complete orthonormal system in H. Let $F \in \mathcal{F}(B^\nu)$ be given by (2.5). Then we have*

(3.8)
$$\int_{B^\nu} F(\rho_1 x_1, \cdots, \rho_\nu x_\nu) dm^\nu(\vec{x})$$

$$= \lim_{n\to\infty}\left[\prod_{j=1}^\nu \rho_j\right]^{-n} \int_{B^\nu} \exp\left\{\sum_{j=1}^\nu \frac{\rho_j^2-1}{2\rho_j^2}\sum_{k=1}^n\left[(e_k,x_j)^\sim\right]^2\right\} F(\vec{x})\, dm^\nu(\vec{x}),$$

where $\displaystyle\prod_{j=1}^{\nu} \rho_j = \rho_1 \rho_2 \cdots \rho_\nu$ and $dm^\nu(\vec{x}) = dm(x_1)dm(x_2)\cdots dm(x_\nu)$.

Proof. Replacing z_j by $\rho_j{}^{-2}$ in the equation (3.4), we have

$$\int_{B^\nu} F(\rho_1 x_1, \cdots, \rho_\nu x_\nu) dm^\nu(\vec{x})$$

$$= \lim_{n\to\infty} \Big[\prod_{j=1}^{\nu} \rho_j\Big]^{-n} \int_{B^\nu} \exp\Big\{ \sum_{j=1}^{\nu} \frac{\rho_j{}^2 - 1}{2\rho_j{}^2} \sum_{k=1}^{n} \big[(e_k, x_j)^\sim\big]^2 \Big\} F(\vec{x})\, dm^\nu(\vec{x})$$

COROLLARY 3.8. *Let $\rho > 0$ and let $\langle e_n \rangle$ be a complete orthonormal system in H. Let F be a functional defined by*

$$F(x) = \int_H \exp\Big\{ i(h, x)^\sim \Big\} d\mu(h), \quad x \in B,$$

for some $\mu \in \mathcal{M}(H)$. Then we have

$$(3.9) \quad \begin{aligned} &\int_B F(\rho x)\, dm(x) \\ &= \lim_{n\to\infty} \rho^{-n} \int_B \exp\Big\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^{n} \big[(e_k, x)^\sim\big]^2 \Big\} F(x)\, dm(x). \end{aligned}$$

Proof. Apply Theorem 3.7 after making the following choices : $\nu = 1$, $\rho_1 = \rho$.

REMARK 3. Corollary 3.8 coincides with Theorem 4 in [10]

The Banach algebra $\mathcal{F}(B^\nu)$ is not closed with respect to pointwise or even uniform convergence [7,p.2], and its uniform closure $\mathrm{Cl}_u(\mathcal{F}(B^\nu))$ with respect to uniform convergence s -a.e. is a larger space than $\mathcal{F}(B^\nu)$. Now we will show that the equation (3.8) also holds for every $F \in \mathrm{Cl}_u(\mathcal{F}(B^\nu))$.

THEORM 3.9. *Let $\rho_k > 0$, $k = 1, 2, \cdots, \nu$ be given and let $\langle e_n \rangle$ be a complete orthonormal system in H. Then the equation (3.8) holds for every $F \in \mathrm{Cl}_u(\mathcal{F}(B^\nu))$.*

Proof. Since $F \in \mathrm{Cl}_u(\mathcal{F}(B^\nu))$, there exists a sequence $\langle F_p \rangle$ from $\mathcal{F}(B^\nu)$ such that $F(\vec{x}) = \lim_{p\to\infty} F_p(\vec{x})$ uniformly s -a.e. on B^ν. Also

since each $F_p \in \mathcal{F}(B^\nu)$, $F_p(\vec{x})$ exists and is bounded s-a.e on B^ν for every $p \in \mathbb{N}$. From the definition of uniform converence s-a.e., it follows that for $\rho_k > 0, k = 1, 2, \cdots, \nu$,

$$(3.10) \qquad F(\rho_1 x_1, \cdots, \rho_\nu x_\nu) = \lim_{p \to \infty} F_p(\rho_1 x_1, \cdots, \rho_\nu x_\nu)$$

uniformly a.e. on B^ν, and

$$(3.11)$$
$$\int_{B^\nu} F(\rho_1 x_1, \cdots, \rho_\nu x_\nu) dm^\nu(\vec{x}) = \lim_{p \to \infty} \int_{B^\nu} F_p(\rho_1 x_1 \cdots, \rho_\nu x_\nu) dm^\nu(\vec{x}).$$

Now taking $z = \rho^{-2}$ and $h = 0$ in the equation (3.3) of Lemma 3.2, we obtain

$$(3.12) \qquad \int_B \exp\left\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^n \left[(e_k, x)^\sim \right]^2 \right\} dm(x) = \rho^n.$$

Since $F(\vec{x}) = \lim_{p \to \infty} F_p(\vec{x})$ uniformly s-a.e. on B^ν, there exist $M > 0$ and a scale-invariant null set N of B^ν such that for all $p \in \mathbb{N}$ and all $\vec{x} \in B^\nu - N$,

$$(3.13) \qquad |F_p(\vec{x})| \leqslant M \text{ and } |F(\vec{x})| \leqslant M.$$

Hence, using (3.12) and (3.13), we obtain

$$\left| \left[\prod_{j=1}^\nu \rho_j \right]^{-n} \int_{B^\nu} \exp\left\{ \sum_{j=1}^\nu \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^n \left[(e_k, x_j)^\sim \right]^2 \right\} F(\vec{x}) \, dm^\nu(\vec{x}) \right.$$
$$\left. - \left[\prod_{j=1}^\nu \rho_j \right]^{-n} \int_{B^\nu} \exp\left\{ \sum_{j=1}^\nu \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^n \left[(e_k, x_j)^\sim \right]^2 \right\} F_p(\vec{x}) \, dm^\nu(\vec{x}) \right|$$
$$\leqslant \left[\prod_{j=1}^\nu \rho_j \right]^{-n} \int_{B^\nu} \exp\left\{ \sum_{j=1}^\nu \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^n \left[(e_k, x_j)^\sim \right]^2 \right\}$$
$$\cdot |F(\vec{x}) - F_p(\vec{x})| \, dm^\nu(\vec{x})$$
$$\leqslant 2M.$$

Finally, using Theorem 3.7, the iterated limit theorem and the dominated convergence theorem, we have

$$
\int_{B^\nu} F(\rho_1 x_1, \cdots, \rho_\nu x_\nu)\, dm^\nu(\vec{x})
$$

$$
= \lim_{p \to \infty} \int_{B^\nu} F_p(\rho_1 x_1, \cdots, \rho_\nu x_\nu)\, dm^\nu(\vec{x})
$$

$$
= \lim_{p \to \infty} \lim_{n \to \infty} \left[\prod_{j=1}^\nu \rho_j\right]^{-n} \int_{B^\nu} \exp\left\{ \sum_{j=1}^\nu \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^n \left[(e_k, x_j)^\sim\right]^2 \right\}
$$
$$
\cdot F_p(\vec{x})\, dm^\nu(\vec{x})
$$

$$
= \lim_{n \to \infty} \lim_{p \to \infty} \left[\prod_{j=1}^\nu \rho_j\right]^{-n} \int_{B^\nu} \exp\left\{ \sum_{j=1}^\nu \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^n \left[(e_k, x_j)^\sim\right]^2 \right\}
$$
$$
\cdot F_p(\vec{x})\, dm^\nu(\vec{x})
$$

$$
= \lim_{n \to \infty} \left[\prod_{j=1}^\nu \rho_j\right]^{-n} \int_{B^\nu} \exp\left\{ \sum_{j=1}^\nu \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^n \left[(e_k, x_j)^\sim\right]^2 \right\}
$$
$$
\cdot F(\vec{x})\, dm^\nu(\vec{x}).
$$

Thus the equation (3.8) holds for every $F \in Cl_u(\mathcal{F}(B^\nu))$

Taking $\nu = 1$ and $\rho_1 = \rho$ in Theorem 3.9, we have the following corollary which cioncides with Theorem 5 in [10]:

COROLLARY 3.10. *Let $\rho > 0$ be given and let $\langle e_n \rangle$ be a complete orthonormal system in H. Then for every $F \in Cl_u(\mathcal{F}(B))$*

$$
\int_B F(\rho x)\, dm(x)
$$
$$
= \lim_{n \to \infty} \rho^{-n} \int_B \exp\left\{ \frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^n \left[(e_k, x)^\sim\right]^2 \right\} F(x)\, dm(x),
$$

where $\mathcal{F}(B)$ is the Fresnel class on B.

Finally, we shall explicitly compute an abstract Wiener integral of certain functional and then check that the equation (3.8) is established. Consider a functional F defined by

$$
(3.14) \qquad F(\vec{x}) = \exp\left\{ \alpha \sum_{j=1}^\nu (h, x_j)^\sim \right\}, \vec{x} = (x_1, \cdots, x_\nu) \in B^\nu.
$$

where h is any fixed element of H and α is a real or complex number.

By the abstract Wiener integration formula, we have

(3.15)

$$\int_{B^\nu} F(\rho_1 x_1, \cdots, \rho_\nu x_\nu)\, dm^\nu(\vec{x}) = \int_{B^\nu} \exp\left\{ \alpha \sum_{j=1}^\nu \rho_j (h, x_j)^\sim \right\} dm^\nu(\vec{x})$$

$$= \prod_{j=1}^\nu \left\{ \int_B \exp\{\alpha \rho_j (h, x_j)^\sim\}\, dm(x_j) \right\} = \exp\left\{ \frac{1}{2}\alpha^2 |h|^2 \sum_{j=1}^\nu \rho_j^2 \right\}.$$

Now let us evaluate the integral in the right hand side of the equation (3.8).

(3.16)

$$\int_{B^\nu} \exp\left\{ \sum_{j=1}^\nu \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^n [(e_k, x_j)^\sim]^2 \right\} \exp\left\{ \alpha \sum_{j=1}^\nu (h, x_j)^\sim \right\} dm^\nu(\vec{x})$$

$$= \prod_{j=1}^\nu \left[\int_B \exp\left\{ \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^n [(e_k, x_j)^\sim]^2 + \alpha \sum_{k=1}^{n+1} c_k (e_k, x_j)^\sim \right\} dm(x_j) \right],$$

where

$$c_k = \begin{cases} \langle e_k, h \rangle, & \text{for } k = 1, 2, \cdots, n. \\ \left[|h|^2 - \sum_{k=1}^n \langle e_k, h \rangle^2 \right]^{\frac{1}{2}}, & \text{for } k = n + 1. \end{cases}$$

The right hand side of (3.16) is equal to the followings :

$$\prod_{j=1}^\nu \left[(2\pi)^{-\frac{(n+1)}{2}} \int_{\mathbf{R}^{n+1}} \exp\left\{ \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^n u_k^2 + \alpha \sum_{k=1}^{n+1} c_k u_k \right\} \right.$$

$$\left. \cdot \exp\left\{ -\frac{1}{2} \sum_{k=1}^{n+1} u_k^2 \right\} du_1 \cdots du_n\, du_{n+1} \right]$$

$$= \prod_{j=1}^\nu \left[(2\pi)^{-\frac{n+1}{2}} \prod_{k=1}^n \left[\int_\mathbf{R} \exp\left\{ -\frac{1}{2\rho_j^2} u_k^2 + \alpha c_k u_k \right\} du_k \right] \right.$$

$$\left. \cdot \int_\mathbf{R} \exp\left\{ -\frac{1}{2} u_{n+1}^2 + \alpha c_{n+1} u_{n+1} \right\} du_{n+1} \right]$$

$$(3.17) \qquad = \left[\prod_{j=1}^{\nu} \rho_j \right]^n \exp\left\{ \frac{\alpha^2}{2} \sum_{j=1}^{\nu} \rho_j^2 \sum_{k=1}^{n} \langle e_k, h \rangle^2 \right\}$$

$$\exp\left\{ \frac{\nu\alpha^2}{2} \left(|h|^2 - \sum_{k=1}^{n} \langle e_k, h \rangle^2 \right) \right\}.$$

Hence

$$(3.18)$$

$$\lim_{n \to \infty} \left[\prod_{j=1}^{\nu} \rho_j \right]^{-n} \int_{B^\nu} \exp\left\{ \sum_{j=1}^{\nu} \frac{\rho_j^2 - 1}{2\rho_j^2} \sum_{k=1}^{n} [(e_k, x_j)^\sim]^2 \right\}$$

$$\cdot \exp\left\{ \alpha \sum_{j=1}^{\nu} (h, x_j)^\sim \right\} dm^\nu(\vec{x})$$

$$= \lim_{n \to \infty} \exp\left\{ \frac{\alpha^2}{2} \sum_{j=1}^{\nu} \rho_j^2 \sum_{k=1}^{n} \langle e_k, h \rangle^2 \right\} \exp\left\{ \frac{\nu\alpha^2}{2} \left(|h|^2 - \sum_{k=1}^{n} \langle e_k, h \rangle^2 \right) \right\}$$

$$= \exp\left\{ \frac{\alpha^2}{2} |h|^2 \sum_{j=1}^{\nu} \rho_j^2 \right\}.$$

Thus we have shown that the equation(3.8) is satisfied for the functional F given in (3.14). In particular, if α is pure imaginary, then F belongs to $\mathcal{F}(B^\nu)$. On the other hand, if $Re(\alpha) \neq 0$, then F is unbounded, so $F \notin \mathcal{F}(B^\nu)$ and also $F \notin Cl_u(\mathcal{F}(B^\nu))$. Thus this example shows that the class of functionals for which the equation (3.8) holds is more extensive than $Cl_u(\mathcal{F}(B^\nu))$.

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Young Sik Kim
Global Analysis Research Center
Department of Mathematics
Seoul Notional University
Seoul 151-742, Korea

Jae Moon Ahn
Department of Mathematics Education
Kon-kuk University
Seoul 133-701, Korea

Kun Soo Chang
Department of Mathematics
Yonsei University
Seoul 120-749, Korea

Il Yoo
Department of Mathematics
Yonsei University
Kangwondo 227-701, Korea