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APPROXIMATE MLE FOR THE SCALE PARAMETER OF THE DOUBLE EXPONENTIAL DISTRIBUTION BASED ON TYPE–II CENSORED SAMPLES

SUK-BOK KANG*

1. Introduction

Consider the double exponential distribution with probability density function (pdf)

$$f(x;\sigma) = \frac{1}{2\sigma} e^{-|x|/\sigma}, \ -\infty < x < \infty. \tag{1.1}$$

Let

$$X_{r+1:n} \le X_{r+2:n} \le \cdots \le X_{n-s:n} \tag{1.2}$$

be the available Type-II censored sample from the double exponential distribution with pdf (1.1), where the smallest r and the largest s observations are censored.

Bain and Engelhardt (1973) discussed the usefulness of the double exponential distribution as a model for statistical studies, and obtained the confidence intervals based on the maximum likelihood estimators for the location and scale parameters of the double exponential distribution. Govindarajulu (1966) gave coefficients of the best linear unbiased estimators for the location and scale parameters in the double exponential distribution from complete and symmetric censored samples. Rao et al.(1991) obtained the first two moments and product

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moments of absolute values of order statistics for the double exponential and the double Weibull distributions. They also obtained the optimum unbiased absolute estimator (OUAE) of the scale parameter in (1.1) by absolute values of the order statistics and showed that this new OUAE is generally more efficient than the best linear unbiased estimator (BLUE) of the scale parameter by order statistics.

The approximate maximum likelihood estimation method was first developed by Balakrishnan (1989 a, b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution and the mean and standard deviation in the normal distribution with censoring. For the extreme value distribution with censoring and the half-logistic distribution with Type-II right censoring, Balakrishnan and Varadan (1991) and Balakrishnan and Wong (1991) provided a method of deriving explicit estimators by approximating the likelihood equation, respectively. They also studied the biases and variances of the proposed estimators and showed that these estimators are almost as efficient as the maximum likelihood estimators and just as efficient as the BLUEs. Some historical remarks and a good summary of the approximate maximum likelihood estimation may be found in Balakrishnan and Cohen (1991).

In this paper, we derive an explicit estimator of σ for complete and censored samples by appropriately approximating the likelihood function. We obtain the asymptotic variance and simulate the values of the biases and the variances of the proposed approximate maximum likelihood estimator (AMLE). We compare the AMLE with the BLUE and the OUAE of the scale parameter σ in the sense of the mean square error (MSE). We show that this proposed AMLE is generally more efficient than the BLUE and the OUAE of the scale parameter for complete and censored samples.

2. Approximate maximum likelihood estimator

We shall derive the AMLE of σ based on the Type-II censored sample in (1.2).

The likelihood function based on the Type-II censored sample in

(1.2) is given by

$$(2.1) \qquad L = \frac{n!}{r!s!}\sigma^{-A}\{F(Z_{r+1:n})\}^r\{1 - F(Z_{n-s:n})\}^s \prod_{i=r+1}^{n-s} f(Z_{i:n})$$

where $A = n - r - s$ is the size of the censored sample in (1.2), $Z_{i:n} = X_{i:n}/\sigma$, and $f(z)$ and $F(z)$ are the pdf and the distribution function of the standard double exponential distribution, respectively. By realizing that $f'(z)/f(z) = -|z|/z, z \neq 0$, we can obtain the following the likelihood equation for σ.

(2.2)

$$\frac{d \ln L}{d\sigma} = -\frac{1}{\sigma}\left[A + rZ_{r+1:n}\frac{f(Z_{r+1:n})}{F(Z_{r+1:n})} - sZ_{n-s:n}\frac{f(Z_{n-s:n})}{1 - F(Z_{n-s:n})} - \sum_{i=r+1}^{n-s} |Z_{i:n}|\right] = 0.$$

Equation (2.2) does not admit an explicit solution for σ unless $r = 0$ and $s = 0$. But we can expand the functions $f(Z_{r+1:n})/F(Z_{r+1:n})$ and $f(Z_{n-s:n})/\{1 - F(Z_{n-s:n})\}$ in Taylor series around the points $F^{-1}(p_{r+1}) = \ln(2p_{r+1})$ if $p_{r+1} < 0.5$, $= -\ln\{2(1-p_{r+1})\}$ if $p_{r+1} \geq 0.5$, and $F^{-1}(p_{n-s}) = \ln(2p_{n-s})$ if $p_{n-s} < 0.5$, $= -\ln\{2(1 - p_{n-s})\}$ if $p_{n-s} \geq 0.5$, respectively.

(1) Case 1 : $z_{r+1:n} \geq 0$

Since $F(Z_{n-s:n}) = 1 - f(Z_{n-s:n})$, the expansion of the function $f(Z_{r+1:n})/F(Z_{r+1:n})$ is required. We can approximate it by

$$(2.3) \qquad \frac{f(Z_{r+1:n})}{F(Z_{r+1:n})} \simeq \alpha - \beta Z_{r+1:n},$$

where

$$p_i = i/(n+1), \qquad q_i = 1 - p_i,$$

$$(2.4) \qquad \alpha = \begin{cases} 1, & p_{r+1} < 0.5 \\ q_{r+1}\{1 - \ln(2q_{r+1})/p_{r+1}\}/p_{r+1}, & p_{r+1} \geq 0.5 \end{cases}$$

$$(2.5) \qquad \beta = \begin{cases} 0, & p_{r+1} < 0.5 \\ q_{r+1}/p_{r+1}^2, & p_{r+1} \geq 0.5 \end{cases}$$

Now making use of the approximate expression in (2.3), we obtain the approximate likelihood equation of (2.2) as

(2.6)

$$\frac{d\ln L}{d\sigma} \simeq \frac{d\ln L^*}{d\sigma}$$
$$= -\frac{1}{\sigma}\left[A + r\alpha Z_{r+1:n} - sZ_{n-s:n} - \sum_{i=r+1}^{n-s} |Z_{i:n}| - r\beta Z_{r+1:n}^2\right] = 0.$$

Upon solving equation (2.6) for σ, we derive the AMLE of σ as

$$\hat{\sigma} = \{B_1 + (B_1^2 + 4AC_1)^{1/2}\}/2A \tag{2.7}$$

where

$$B_1 = sX_{n-s:n} + \sum_{i=r+1}^{n-s} |X_{i:n}| - r\alpha X_{r+1:n},$$
$$C_1 = r\beta X_{r+1:n}^2.$$

Equation (2.6) is quadratic in σ which has two roots; however, one of them drops out because $C_1 \geq 0$.

Case 2 : $z_{r+1} < 0 < z_{n-s:n}$

Since $F(Z_{r+1}) = f(Z_{r+1})$ and $F(Z_{n-s}) = 1 - f(Z_{n-s})$ in this case, we can obtain the likelihood equation as follow;

$$\frac{d\ln L}{d\sigma} = -\frac{1}{\sigma}\left[A + rZ_{r+1:n} - sZ_{n-s:n} - \sum_{i=r+1}^{n-s} |Z_{i:n}|\right] = 0. \tag{2.8}$$

Hence, in this case, we can obtain the exact maximum likelihood estimator (MLE) of σ as follow;

$$\hat{\sigma} = \{sX_{n-s:n} - rX_{r+1:n} + \sum_{i=r+1}^{n-s} |X_{i:n}|\}/A, \tag{2.9}$$

Case 3 : $z_{n-s:n} \leq 0$

Since $F(Z_{r+1:n}) = f(Z_{r+1:n})$, the expansion of the function

$f(Z_{n-s:n})/\{1-F(Z_{n-s:n})\}$ is required. We can approximate it by

$$\text{(2.10)} \qquad \frac{f(Z_{n-s:n})}{\{1-F(Z_{n-s:n})\}} \simeq \gamma + \delta Z_{n-s:n},$$

where

$$\text{(2.11)} \qquad \gamma = \begin{cases} 1, & p_{r+1} > 0.5 \\ p_{r+1}\{1-\ln(2p_{r+1})/q_{r+1}\}/q_{r+1}, & p_{r+1} \leq 0.5 \end{cases}$$

$$\text{(2.12)} \qquad \delta = \begin{cases} 0, & p_{r+1} > 0.5 \\ p_{r+1}/q_{r+1}^2, & p_{r+1} \leq 0.5 \end{cases}$$

Now making use of the approximate expression in (2.10), we can obtain the approximate likelihood equation of (2.2) as

(2.13)

$$\frac{d\ln L}{d\sigma} \simeq \frac{d\ln L^*}{d\sigma} = -\frac{1}{\sigma}\left[A + rZ_{r+1:n} - s\gamma Z_{n-s:n} - \sum_{i=r+1}^{n-s} |Z_{i:n}| - s\delta Z_{n-s:n}^2\right] = 0.$$

Upon solving equation (2.13) for σ, we derive the AMLE of σ as

$$\text{(2.14)} \qquad \hat{\sigma} = \{B_2 + (B_2^2 + 4AC_2)^{1/2}\}/2A,$$

where

$$B_2 = s\gamma X_{n-s:n} + \sum_{i=r+1}^{n-s} |X_{i:n}| - rX_{r+1:n},$$

$$C_2 = s\delta X_{n-s:n}^2.$$

Equation (2.13) is quadratic in σ which has two roots; however, one of them drops out because $C_2 \geq 0$.

Note that in complete case ($r = s = 0$), the AMLE becomes the minimum variance unbiased estimator, $\sum_{i=1}^{n} |X_{i:n}|/n$ of σ.

3. Asymptotic Properties

Since the AMLEs $\hat{\sigma}$ in (2.7) and (2.14) are the solutions of the approximate maximum likelihood equations (2.6) and (2.13), respectively, and the MLE $\hat{\sigma}$ in (2.9) is the solution of the maximum likelihood equation (2.8), it immediately follows that $\hat{\sigma}$ is asymptotically normally distributed with mean σ and variance $1/E\{-d^2 \ln L^*/d\sigma^2\}$ (See Kendall and Stuart (1973)). Now, from equations (2.6), (2.8), and (2.13) we can obtain

$$(3.1) \qquad E\Big(-\frac{d^2 \ln L^*}{d\sigma^2}\Big) = \begin{cases} D_1/\sigma^2, & z_{r+1:n} \geq 0 \\ D_2/\sigma^2, & z_{r+1:n} < 0 < z_{n-s:n} \\ D_3/\sigma^2, & z_{n-s:n} \leq 0 \end{cases}$$

where

$$\begin{aligned} D_1 &= 3r\beta E(Z_{r+1:n}^2) - 2\{r\alpha E(Z_{r+1:n}) - sE(Z_{n-s:n}) \\ &\quad - \sum_{i=r+1}^{n-s} E(|Z_{i:n})|\} - A, \\ D_2 &= 2\{sE(Z_{n-s:n}) + \sum_{i=r+1}^{n-s} E(|Z_{i:n}|) - rE(Z_{r+1:n})\} - A, \\ D_3 &= 3s\delta E(Z_{n-s:n}^2) - 2\{rE(Z_{r+1:n}) - s\gamma E(Z_{n-s:n}) \\ &\quad - \sum_{i=r+1}^{n-s} E(|Z_{i:n}|)\} - A. \end{aligned}$$

From the equation (3.1), we can compute the asymptotic variance of the AMLE $\hat{\sigma}$ by using the following results (Govindarajulu (1966) and Rao et al.(1991)).

$$E(Z_{i:n}) = 2^{-n}\left\{\sum_{j=0}^{i-1}\binom{n}{j}S_1(i-j, n-j) - \sum_{j=i}^{n}\binom{n}{j}S_1(j-i+1, j)\right\}$$

$$E(|Z_{i:n}|) = 2^{-n}\left\{\sum_{j=0}^{i-1}\binom{n}{j}S_1(i-j,n-j)+\sum_{j=i}^{n}\binom{n}{j}S_1(j-i+1,j)\right\}$$

$$E(Z_{i:n}^2) = 2^{-n}\left\{\sum_{j=0}^{i-1}\binom{n}{j}[S_2(i-j,n-j)+S_1^2(i-j,n-j)]\right.$$

$$\left.+\sum_{j=i}^{n}\binom{n}{j}[S_2(j-i+1,j)+S_1^2(j-i+1,j)]\right\},$$

$$i = r+1,\dots,n-s,$$

where

$$S_k(i,n) = \sum_{l=n-i+1}^{n} 1/l^k, \qquad k = 1,2.$$

From the formulae derived in section 2, we simulate the values of the biases, variances and the mean square errors of AMLE $\hat{\sigma}$ (based on 10000 Monte Carlo runs) for $n = 3(1)10$, 20, 30 and over various choices of censoring. These values are presented in Table 1, 2 and 3 with the variances of the BLUE σ^* and the OUAE $\tilde{\sigma}$. The values of the relative efficiency of the AMLE $\hat{\sigma}$ to the BLUE σ^*, REF$(\hat{\sigma},\sigma^*) =$ Var(σ^*)/MSE$(\hat{\sigma})$, and the relative efficiency of the AMLE $\hat{\sigma}$ to the OUAE $\tilde{\sigma}$, REF$(\hat{\sigma},\tilde{\sigma}) =$ Var$(\tilde{\sigma})$/MSE$(\hat{\sigma})$ are also presented in Table 1 and 2. From Table 1 and 2, the AMLE is generally more efficient than the BLUE and OUAE of the double exponential scale parameter σ for complete, symmetrically censored samples and right censored samples.

Table 1. The Biases and MSE's of the AMLE $\hat{\sigma}$, and the Variances of the BLUE σ^* and OUAE $\tilde{\sigma}$ of the Double Exponential Scale Parameter from Complete and Symmetrically Censored Samples.

n	$r=s$	$\frac{1}{\sigma}$Bias($\hat{\sigma}$)	$\frac{1}{\sigma^2}$MSE($\hat{\sigma}$)	$\frac{1}{\sigma^2}$Var(σ^*)	$\frac{1}{\sigma^2}$Var($\tilde{\sigma}$)	REF($\hat{\sigma},\sigma^*$)	REF($\hat{\sigma},\tilde{\sigma}$)
3	0	0.0133	0.3517	0.4321	0.3333	1.2286	0.9477
4	0	-0.0064	0.2433	0.2986	0.2500	1.2273	1.0275
	1	-0.0831	0.4352	0.8512	0.5102	1.9559	1.1723
5	0	0.0011	0.1990	0.2290	0.2000	1.1508	1.0050
	1	-0.0274	0.3180	0.4387	0.3367	1.3796	1.0588
6	0	0.0001	0.1654	0.1858	0.1667	1.1233	1.0079
	1	-0.0070	0.2435	0.2996	0.2510	1.2304	1.0308
	2	-0.1189	0.4209	0.8866	0.5299	2.1064	1.2590
7	0	0.0078	0.1474	0.1565	0.1429	1.0617	0.9695
	1	-0.0018	0.2007	0.2288	0.2003	1.1400	0.9980
	2	-0.0312	0.3146	0.4468	0.3473	1.4202	1.1039
8	0	0.0038	0.1249	0.1351	0.1250	1.0817	1.0008
	1	-0.0012	0.1679	0.1856	0.1668	1.1054	0.9934
	2	-0.0194	0.2419	0.3020	0.2551	1.2484	1.0546
	3	-0.1637	0.3959	0.9078	0.5453	2.2930	1.3774
9	0	-0.0004	0.1105	0.1190	0.1111	1.0769	1.0054
	1	0.0013	0.1432	0.1562	0.1429	1.0908	0.9979
	2	-0.0063	0.1991	0.2295	0.2019	1.1527	1.0141
	3	-0.0593	0.3030	0.4534	0.3603	1.4964	1.1891
10	0	-0.0031	0.0997	0.1062	0.1000	1.0652	1.0030
	1	-0.0001	0.1263	0.1350	0.1250	1.0689	0.9897
	2	0.0003	0.1649	0.1857	0.1674	1.1261	1.0152
	3	-0.0177	0.2405	0.3044	0.2613	1.2657	1.0865

Table 2. The Biases and MSE's of the AMLE $\hat{\sigma}$, and the Variances of the BLUE σ^* and OUAE $\tilde{\sigma}$ of the Double Exponential Scale Parameter from Right Censored Samples.

n	s	$\frac{1}{\sigma}$Bias($\hat{\sigma}$)	$\frac{1}{\sigma^2}$ MSE($\hat{\sigma}$)	$\frac{1}{\sigma^2}$Var(σ^*)	$\frac{1}{\sigma^2}$Var($\tilde{\sigma}$)	REF($\hat{\sigma},\sigma^*$)	REF($\hat{\sigma},\tilde{\sigma}$)
3	1	-0.1159	0.4272	0.8422	0.5290	1.9714	1.2383
4	1	-0.0290	0.3146	0.4414	0.3391	1.4031	1.0779
	2	0.0693	0.5508	0.7343	0.5484	1.3332	0.9956
5	1	-0.0041	0.2440	0.3008	0.2514	1.2328	1.0303
	2	-0.0916	0.3098	0.4483	0.3657	1.4471	1.1804
	3	0.0638	0.4867	0.5800	0.4935	1.1917	1.0140
6	1	-0.0001	0.2051	0.2294	0.2004	1.1185	0.9771
	2	-0.0229	0.2363	0.3063	0.2612	1.2962	1.1054
	3	0.0573	0.3551	0.4245	0.3775	1.1954	1.0631
	4	0.0466	0.4163	0.4668	0.4274	1.1213	1.0267
7	1	-0.0114	0.1614	0.1858	0.1668	1.1512	1.0335
	2	-0.0094	0.1970	0.2316	0.2039	1.1756	1.0350
	3	-0.0712	0.2346	0.3075	0.2791	1.3107	1.1897
	4	0.0405	0.3185	0.3802	0.3537	1.1937	1.1105
	5	0.0357	0.3752	0.3897	0.3714	1.0386	0.9899
8	1	-0.0017	0.1238	0.1564	0.1429	1.2633	1.1543
	2	-0.0027	0.1667	0.1867	0.1681	1.1200	1.0084
	3	-0.0339	0.1942	0.2343	0.2132	1.2065	1.0978
	4	0.0337	0.2527	0.2982	0.2868	1.1801	1.1349
	5	0.0399	0.3011	0.3345	0.3185	1.1109	1.0578
	6	0.0279	0.3277	0.3354	0.3268	1.0235	0.9973
9	1	-0.0082	0.1221	0.1351	0.1250	1.1065	1.0238
	2	-0.0004	0.1383	0.1567	0.1434	1.1330	1.0369
	3	-0.0084	0.1645	0.1883	0.1723	1.1447	1.0474
	4	-0.0658	0.1863	0.2345	0.2254	1.2587	1.2099
	5	0.0346	0.2415	0.2798	0.2750	1.1586	1.1387
	6	0.0341	0.2751	0.2952	0.2856	1.0731	1.0382
	7	0.0288	0.2934	0.2954	0.2914	1.0068	0.9932
10	1	-0.0039	0.1097	0.1189	0.1111	1.0839	1.0128
	2	0.0053	0.1269	0.1352	0.1252	1.0654	0.9866
	3	-0.0052	0.1427	0.1575	0.1453	1.1037	1.0182
	4	-0.0367	0.1559	0.1897	0.1802	1.2168	1.1559
	5	0.0299	0.2003	0.2298	0.2307	1.1473	1.1518
	6	0.0264	0.2331	0.2576	0.2541	1.1051	1.0901
	7	0.0087	0.2425	0.2632	0.2576	1.0854	1.0623
	8	0.0236	0.2709	0.2649	0.2630	0.9779	0.9708

Table 3. The Biases, Variances and the Asymptotic Variances of the AMLE $\hat{\sigma}$ of the Double Exponential Scale Parameter from Type-II Censored Samples.

		$n = 10$		$n = 20$			$n = 30$		
r	s	Bias	VAR	Bias	VAR	AVAR	Bias	VAR	AVAR
0	1	-0.0008	0.1126	-0.0008	0.0525	0.0526	0.0012	0.0342	0.0345
0	2	-0.0010	0.1206	0.0001	0.0548	0.0556	0.0005	0.0357	0.0357
0	3	0.0030	0.1434	-0.0038	0.0584	0.0588	-0.0020	0.0370	0.0370
0	4	-0.0252	0.1640	0.0028	0.0646	0.0625	-0.0007	0.0382	0.0385
1	0	0.0051	0.1136	-0.0012	0.0512	0.0526	0.0006	0.0346	0.0345
1	1	0.0022	0.1240	-0.0017	0.0547	0.0556	0.0000	0.0351	0.0357
1	2	0.0067	0.1430	-0.0021	0.0596	0.0588	0.0010	0.0364	0.0370
1	3	-0.0077	0.1636	-0.0026	0.0634	0.0625	-0.0008	0.0386	0.0385
1	4	-0.0315	0.1919	0.0024	0.0663	0.0667	0.0036	0.0398	0.0400
2	0	-0.0029	0.1248	-0.0014	0.0552	0.0556	-0.0027	0.0350	0.0357
2	1	-0.0062	0.1454	-0.0005	0.0579	0.0588	-0.0021	0.0375	0.0370
2	2	-0.0038	0.1717	-0.0015	0.0624	0.0625	-0.0014	0.0377	0.0385
2	3	-0.0088	0.1963	-0.0033	0.0668	0.0667	-0.0008	0.0393	0.0400
2	4	-0.0570	0.2326	0.0004	0.0724	0.0714	-0.0020	0.0406	0.0417
3	0	-0.0054	0.1356	0.0032	0.0608	0.0588	-0.0007	0.0378	0.0370
3	1	-0.0098	0.1637	0.0026	0.0636	0.0625	-0.0004	0.0380	0.0385
3	2	-0.0106	0.1940	-0.0006	0.0678	0.0667	0.0014	0.0409	0.0400
3	3	-0.0277	0.2343	-0.0020	0.0700	0.0714	-0.0027	0.0404	0.0417
3	4	-0.0734	0.2991	0.0018	0.0781	0.0769	-0.0002	0.0430	0.0435
4	0	-0.0289	0.1616	-0.0028	0.0623	0.0625	-0.0016	0.0384	0.0385
4	1	-0.0393	0.1894	-0.0016	0.0641	0.0667	0.0027	0.0408	0.0400
4	2	-0.0577	0.2299	0.0024	0.0713	0.0714	0.0004	0.0417	0.0417
4	3	-0.0798	0.2973	0.0014	0.0780	0.0769	0.0026	0.0445	0.0435
4	4	-0.1826	0.3644	-0.0031	0.0827	0.0833	-0.0025	0.0452	0.0455

Bias:$=E(\hat{\sigma} - \sigma)/\sigma$, VAR:$=Var(\hat{\sigma})/\sigma^2$, AVAR:$=1/E(-d^2 \ln L^*/d\sigma^2)/\sigma^2$

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Department of Statistics
Yeungnam University
Kyongsan 712-749, Korea