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DYNAMICS OF TWO PARAMETER FAMILY OF POINT–MASS SINGULAR INNER FUNCTIONS

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1. Introduction

Let $\alpha > 0$ and ζ be a point on the unit circle $T = \{|z| = 1\}$ of the complex plane C. The point-mass singular inner functin

$$M_{\zeta,\alpha}(z) = \exp\left(-\alpha\frac{\zeta + z}{\zeta - z}\right)$$

of point-mass α at ζ is analytic on C except for the singularity at ζ. We previously determined the integral representation of the interates of $M_{\zeta,\alpha}$, the location of its Denjoy-Wolff points and ergodic property of $M_{\zeta,\alpha}$ for certain cases in [4,5]. We continue the study of the dynamical behaviour of the family. We determine the Julia sets and the local dynamical behaviour at the indifferent Denjoy-Wolff points of $M_{\zeta,\alpha}$, which occur in the case $0 < \alpha \leq 2$ and $\zeta = e^{i\phi}$ with $\phi = \pm(\sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2})$. To avoid the lengthy preliminaries, we refer to [1,2,3,6] for the basic theory and terminology for the complex dynamics.

2. Dynamics at the indifferent Denjoy-Wolff points

The Denjoy-Wolff point of $M_{\zeta,\alpha}$ is the unique fixed point δ in $|z| \leq 1$ that is attractive or indifferent (i.e., its multiplier $M'_{\zeta,\alpha}(\delta)$ is of modulus less than or equal to one, respectively.) Its location and multiplier play an important role in the dynamics of the function. In this section, we describe the petal structures at the indifferent Denjoy-Wolff point of $M_{\zeta,\alpha}$. The following proposition is a more precise version of Theorem 3.1 of [5] and its proof.

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2.1. THEOREM. *Let $\zeta = e^{i\phi}$, where $0 \le |\phi| \le \pi$.*

(a) *If $\alpha > 2$, then the Denjoy-Wolff point of $M_{\zeta,\alpha}$ is attractive and lies in the open unit disc $U = \{|z| < 1\}$ for any ζ.*
(b) *If $0 < \alpha \le 2$ and if $|\phi| < \sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2}$, then the Denjoy-Wolff point of $M_{\zeta,\alpha}$ is also attractive and lies in U.*
(c) *If $0 < \alpha \le 2$ and if $\phi = \pm(\sqrt{\alpha(2-\alpha)}+2\sin^{-1}\sqrt{\alpha/2})$, then the Denjoy-Wolff point of $M_{\zeta,\alpha}$ is indifferent and lies at $e^{\pm i\sqrt{\alpha(2-\alpha)}}$ on the unit circle T, respectively.*
(d) *If $0 < \alpha < 2$ and if $\sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2} < |\phi| \le \pi$, then the Denjoy-Wolff point of $M_{\zeta,\alpha}$ is attractive and lies on T.*

Let $0 < \alpha \le 2$ and $\phi = \pm(\sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2})$. Then $M_{\zeta,\alpha}$ has its indifferent Denjoy-Wolff point at $\delta = e^{\pm i\sqrt{\alpha(2-\alpha)}}$, respectively. To describe the local dynamics at δ, we conjugate $M_{\zeta,\alpha}$ by the linear fractional transformation

$$T(z) = -e^{\pm i\sqrt{\alpha(2-\alpha)}}\frac{z-i}{z+i}$$

to get its conjugate function

$$\begin{aligned} f(z) =& T^{-1} \circ M_{\zeta,\alpha} \circ T(z) \\ =& \tan\left(\frac{z}{1 \mp \sqrt{\frac{2-\alpha}{\alpha}}z}\right) \\ =& z \pm \sqrt{\frac{2-\alpha}{\alpha}}z^2 + \frac{6-2\alpha}{3\alpha}z^3 \pm \frac{2}{\alpha}\sqrt{\frac{2-\alpha}{\alpha}}z^4 + \cdots, \end{aligned}$$

by some routine algebra.

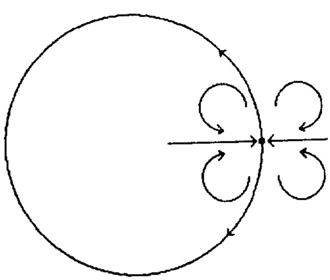
The local dynamics of $M_{\zeta,\alpha}$ at the indifferent Denjoy-Wolff point $\delta = e^{\pm i\sqrt{\alpha(2-\alpha)}}$ can be deduced from that of $f(z)$ at its indifferent fixed point $z = 0$. We have the following theorem by applying the Petal Theorem [1] to the conjugate function f.

2.2. THEOREM. *Let $0 < \alpha \le 2$ and let $\phi = \pm(\sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2})$.*

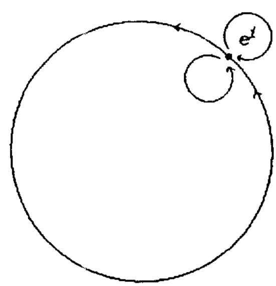
(a) *If $\alpha = 2$ (and so $\phi = \pi$, i.e., $\zeta = -1$), then $M_{\zeta,\alpha}$ has two attracting petals and two repelling petals at the Denjoy-Wolff point at $\delta = 1$.*

(b) *If* $0 < \alpha < 2$, *then* $M_{\zeta,\alpha}$ *has one attracting petal and one repelling petal at the Denjoy-Wolff point* $e^{\pm i\sqrt{\alpha(2-\alpha)}}$, *respectively. In this case,* $M^n_{\zeta,\alpha}(0)$ *converges to the Denjoy-Wolff point* $e^{\pm i\sqrt{\alpha(2-\alpha)}}$ *tangentially as* $n \to \infty$, *respectively.*

Proof. (a) If $\alpha = 2$, then $f(z) = z + \frac{1}{3}z^3 + \cdots$. By the Petal Theorem [1], f has two attracting petals and two repelling petals at the indifferent fixed point $z = 0$. Therefore, $M_{\zeta,\alpha}$ has the same property at $\delta = 1$. The local dynamics at $\delta = 1$ is described in Figure 1(a).



(a) $\alpha = 2$



(b) $0 < \alpha = 1 < 2$

Figure 1

(b) If $0 < \alpha < 2$, then $f(z) = z \pm \sqrt{\frac{2-\alpha}{\alpha}}z^2 + \cdots$. Therefore, it has one attracting petal and one repelling petal at the indifferent fixed point at $z = 0$, and so does $M_{\zeta,\alpha}$ at its indifferent Denjoy-Wolff point $\delta = e^{\pm i\sqrt{\alpha(2-\alpha)}}$. In this case, $M^n_{\zeta,\alpha}(0)$ converges to its Denjoy-Wolff point $\delta = e^{\pm i\sqrt{\alpha(2-\alpha)}}$ tangentially as $n \to \infty$, depending on $\phi = \pm(\sqrt{\alpha(2-\alpha)} + \sin^{-1}\sqrt{\alpha/2})$, respectively, by the Petal Theorem. The local dynamics at $\delta = e^{i\sqrt{\alpha(2-\alpha)}}$ is described in Figure 1(b).

3. Julia Sets

We recall that the Julia sets of an analytic function f is the set of points that have no neighborhood on which the family of the iterates $\{f^n\}$ is a normal family in the sense of Montel. It is also well known that the Julia set is given as the boundary of the basin of attraction of any attractive cycle. The function $M_{\zeta,\alpha}$ has an essential singularity at

ζ and the family of iterates $\{M^n_{\zeta,\alpha}\}$ cannot be a normal family on any neighborhood of ζ. The essential singular point ζ and its preimages under $M_{\zeta,\alpha}$ are assumed to belong the Julia set With this understanding, we have the following characterization of the Julia sets of the family $M_{\zeta,\alpha}$.

3.1. THEOREM. *Let $\alpha > 0$ and let $\zeta = e^{i\phi}$ with $0 \leq$*

(a) *If $\alpha > 2$ or if $0 < \alpha \leq 2$ and $|\phi| < \sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2}$, then the Julia set of $M_{\zeta,\alpha}$ is the unit circle T.*

(b) *If $\alpha = 2$ and $\phi = \pi$, i.e., $\zeta = -1$, then the Julia set of $M_{\zeta,\alpha}$ is also T.*

(c) *If $0 < \alpha < 2$ and $\sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2} \leq |\phi| \leq \pi$, then the Julia set of $M_{\zeta,\alpha}$ is a Cantor set on the unit circle T, i.e., a closed, totally disconnected, perfect subset of the unit circle.*

The following proposition is easily derived from the property of the Denjoy-Wolff point.

3.2. PROPOSITION. *If δ is the Denjoy-Wolff point of $M_{\zeta,\alpha}$, then $M^n_{\zeta,\alpha}(z) \to \delta/|\delta|^2$ as $n \to \infty$ for $|z| > 1$.*

Proof. Let $|z| > 1$. Then

$$\begin{aligned}
1/\overline{M_{\zeta,\alpha}(1/\overline{z})} &= \exp\left(\alpha\frac{\overline{\zeta} + 1/\overline{z}}{\overline{\zeta} - 1/\overline{z}}\right) \\
&= \exp\left(-\alpha\frac{\zeta + z}{\zeta - z}\right) \\
&= M_{\zeta,\alpha}(z).
\end{aligned}$$

Therefore,

$$M^2_{\zeta,\alpha}(z) = M_{\zeta,\alpha}(1/\overline{M_{\zeta,\alpha}(1/\overline{z})})1/\overline{M^2_{\zeta,\alpha}(1/\overline{z})}).$$

We have, by induction,

$$M^n_{\zeta,\alpha}(z) = 1/\overline{M^n_{\zeta,\alpha}(1/\overline{z})}.$$

Since $|1/\overline{z}| < 1$, we have $M^n_{\zeta,\alpha}(1/\overline{z}) \to \delta$ as $n \to \infty$ and so

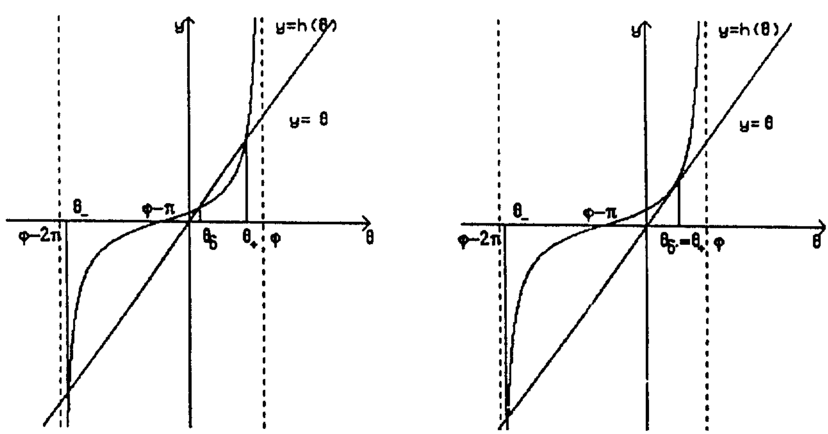
$$M^n_{\zeta,\alpha}(z) \to 1/\overline{\delta} = \delta/|\delta|^2,$$

as $n \to \infty$. Now, we prove Theorem 3.1.

3.3. *Proof of Theorem 3.1.* **(a)** In this case, the Denjoy-Wolff point δ of $M_{\zeta,\alpha}$ lies in the open unit disc U. For $|z| \neq 1$, by Proposition 3.2, $M^n_{\zeta,\alpha}(z) \to \delta$, or $\delta/|\delta^2|$ as $n \to \infty$ depending on whether $|z| < 1$ or $|z| > 1$, respectively. Therefore $\{M^n_{\zeta,\alpha}\}$ cannot be a normal family on any neighborhood of the points on T and so the Julia set is T.
(b) In this case, the Denjoy-Wolff point is $\delta = 1$. The local dynamics at $\delta = 1$ is described in Figure 1(a). For $|z| \neq 1, M^n_{\zeta,\alpha}(z) \to 1$, as $n \to \infty$ by Proposition 3.2. The restriction of $M_{-1,2}$ on the unit circle has the form

$$M_{-1,2}(e^{i\theta}) = \exp\left(2i\tan\frac{\theta}{2}\right),$$

which has a repelling fixed point at $z = 1$ as shown in Figure 1(a). Therefore $\{M^n_{\zeta,\alpha}\}$ cannot be a normal family on any neighborhood of the points on the unit circle. The unit circle is again the Julia set of $M_{-1,2}$.



(a) $\sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2} < \phi \leq \pi$ (b) $\phi = \sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2}$

Figure 2

(c) We only consider the case $\sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2} \leq \phi \leq \pi$. The remaining case is similarly proved. The Denjoy-Wolff point δ in this case is on the unit circle. Since any point not on the unit circle belongs to the basin of attraction of δ by the Proposition 3.2, we consider the dynamical behavior of $M_{\zeta,\alpha}$ on the unit circle. Let $\zeta = e^{i\phi}$ and $z = e^{i\theta}$. We note that

$$M_{\zeta,\alpha}(e^{i\theta}) = \exp\left(-i\alpha \cot \frac{\theta - \phi}{2}\right) = \exp\left(i\alpha \tan \frac{\theta + \pi - \phi}{2}\right).$$

The local behavior of $M_{\zeta,\alpha}$ near δ can be deduced from the consideration of the function

$$h(\theta) = \alpha \tan \frac{\theta + \pi - \phi}{2}$$

which has three fixed points in the interval $(\phi - 2\pi, \phi) : \theta_- < \theta_\delta \le \theta_+$, in magnitude, as shown in Figure 2. We note that θ_δ corresponds to the Denjoy-Wolff point δ as $\delta = e^{i\theta_\delta}$ and it is equal to θ_+ in the case $\phi = \sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2}$. The graphical analysis of Figure 2 shows that every point $z = e^{i\theta}$ with $\theta_- < \theta < \theta_+$ is attracted to the Denjoy-Wolff point $\delta = e^{i\theta_\delta}$ under the iteration of $M_{\zeta,\alpha}$. Let $A^{(0)}$ be the arc on the unit circle T corresponding to the interval (θ_-, θ_+) and $B^{(0)}$ be the complementary closed arc on the unit circle. We also write $A^{(n)} = M_{\zeta,\alpha}^{-n}(A^{(0)})$, $n = 1, 2, \cdots$ and will show that the Julia set $J_{\zeta,\alpha}$ of $M_{\zeta,\alpha}$ is given by

$$J_{\zeta,\alpha} = T \setminus \cup_{n=0}^{\infty} A^{(n)} \tag{1}$$

and it is a Cantor set. For example, $A^{(1)}$ consists of infinitely many open subarcs of $B^{(0)}$, whose closures are disjoint, which correspond to the open subintervals of $(\phi - 2\pi, \phi)$ which are mapped to the intervals $(\theta_- + 2k\pi, \theta_+ + 2k\pi)$, $k = \pm 1, \pm 2, \cdots$ by the map h. $A^{(2)}$ consists of infinitely many open subarcs of the complement of $A^{(0)} \cup A^{(1)}$, whose closures are disjoint, which correspond to the open subintervals of $(\phi - 2\pi, \phi)$ which are mapped to the intervals $(\theta_- + 2k\pi, \theta_+ + 2k\pi)$, $k = \pm 1, \pm 2, \cdots$, by map h^2, and so on. From the construction, $\cup_{n=0}^{\infty} A^{(n)}$ is the basin of attraction of $M_{\zeta,\alpha}$ restricted on T. Therefore, the Julia set of $M_{\zeta,\alpha}$ is given by (eq1). It is clearly closed and invariant under $M_{\zeta,\alpha}$. To show that it is totally disconnected, we suppose that a nondegenerate arc I is contained in $J_{\zeta,\alpha}$. Then $M_{\zeta,\alpha}^n(I)$, $n = 1, 2, \cdots$ is an arc contained in $B^{(0)}$

whose length is getting arbitrarily bigger as n tends to ∞, which is impossible. Hence $J_{\zeta,\alpha}$ is totally disconnected. To show that it is a perfect set, we suppose that it has an isolated point. Then there should be an open arc I about η so that $I\backslash\{\eta\} \subset \cup_{n=0}^{\infty} A^{(n)}$. From the construction of $A^{(n)}$, each of the two open subarcs of $I\backslash\{\eta\}$ should be contained in an open subarcs comprising of $A^{(n)}$ for some n. Therefore $M_{\zeta,\alpha}^{n}(I\backslash\{\eta\})$ is contained in $A^{(0)}$ and consists of two open subarcs of $A^{(0)}$ with a common boundary point η. Hence $M_{\zeta,\alpha}^{n}(\eta)$ should belong to $A^{(0)}$, which is a contradiction. Therefore, $J_{\zeta,\alpha}$ does not contain an isolated point and so it is a perfect set. This completes the proof.

4. Parameter regions for connected Julia sets

Finally, we describe the parameter region for connected Julia sets in Figure 3. If we write $w = \alpha\zeta = \alpha e^{i\phi}$ for $\alpha > 0$ and $\zeta = e^{i\phi}$, then the region K of the complex parameters $w = \alpha\zeta$ for which $M_{\zeta,\alpha}$'s have connected Julia sets is given as

$$K = \Big\{\alpha e^{i\phi} : \alpha \geq 2, \quad \text{or} \quad 0 < \alpha < 2 \ \text{ and } \ |\phi| < \sqrt{\alpha(2-\alpha)} + 2\sin^{-1}\sqrt{\alpha/2}\Big\}$$

by Theorem 3.1 and it is represented by the shaded region in Figure 3.

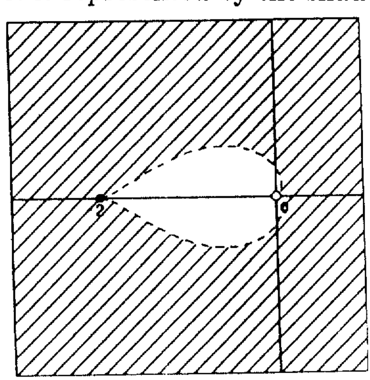


Figure 3

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