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ON THE SPECTRAL MAXIMAL SPACES OF A MULTIPLICATION OPERATOR

JAE CHUL RHO AND JONG-KWANG YOO

1. Introduction

In [13], Pták and Vrbová proved that if T is a bounded normal operator T on a complex Hilbert space H, then the ranges of the spectral projections can be represented in the form

$$\mathcal{E}(F)H = \bigcap_{\lambda \notin F} (T - \lambda I)H \quad \text{for all closed subsets } F \text{ of } \mathbb{C},$$

where $\mathcal{E}$ denotes the spectral measure associated with T. The algebraic representation of the spaces $\mathcal{E}(F)H$ turned out to be useful for the automatic continuity of linear transformations intertwining a given pair of certain decomposable operators. For applications in automatic continuity theory, we refer to [3], [8],[11],[12],[18].

In the present paper, we show that a theorem describing structure of certain class of invariant subspaces of multiplication operators, so called spectral maximal spaces(terms to be defined below). And we attempt to give a simple and unified approach to the algebraic representation of the spectral maximal spaces which is an analogy of the normal operators, i.e., if A is a semi-simple commutative Banach algebra and T_a has the weak-2 spectral decomposition property then there exists an idempotent element $r \in A$ such that

$$T_r(A) = \bigcap_{\lambda \notin F} (T_a - \lambda I)A \quad \text{for all closed subsets } F \text{ of } \mathbb{C},$$

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where $T_r x := rx$ for all $x \in A$.

We first recall some definitions and known results concerning local spectral theory included in [5]. Given a complex Banach space X, $\mathcal{L}(X)$ denotes the Banach algebra of all bounded linear operators on X. Given an operator $T \in \mathcal{L}(X)$, let $Lat(T)$ stands for the collection of all closed linear subspaces of X which are invariant under T. Recall that the notion of local spectrum from [5]; if $x \in X$ then the *local spectrum* $\sigma_T(x)$ of T at x is defined to be the complement of the set of all $\lambda \in \mathbb{C}$ for which there is analytic $f : U \to X$ on some open neighborhood U of λ such that $(T - \mu I)f(\mu) = x$ for all $\mu \in U$. A *spectral maximal space* of T is an invariant subspace Y of T, which contains any invariant subspace Z with the property $\sigma(T|Z) \subseteq \sigma(T|Y)$. An operator $T \in \mathcal{L}(X)$ is called *decomposable* (resp, *weak-2 spectral decomposition property*) if for every open cover $\{U, V\}$ of $\mathbb{C}$, there exists $Y, Z \in Lat(T)$ such that $X = Y + Z$ $(resp,\ X = \overline{Y + Z}\ )$, $\sigma(T|Y) \subseteq U$ and $\sigma(T|Z) \subseteq V$.

It follows from the example given by E. Albrecht [2] that in general, weak-2 spectral decomposition property is strictly weaker than decomposability. An important feature in this theory of the decomposable operator is the investigation of the spectral maximal spaces given by $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ for all closed subset F of $\mathbb{C}$. If $T \in \mathcal{L}(X)$ has the single-valued extension property, then for each closed $F \subseteq \mathbb{C}$, $X_T(F)$ is a linear subspace (not necessarily closed) of X, hyperinvariant for T. Let T be a linear operator on a complex Banach space X. We say that a subspace Y of X is *T-divisible* if $(T - \lambda I)Y = Y$ for each $\lambda \in \mathbb{C}$.

Consider the class of all linear subspaces Y in X which satisfy $(T - \lambda I)Y = Y$ for all $\lambda \in \mathbb{C} \setminus F$ and set $E_T(F) := span Y$. It is obvious that $(T - \lambda I)E_T(F) = E_T(F)$ for $\lambda \in \mathbb{C} \setminus F$ as well so that the class which we consider has a maximal element if ordered by inclusion. In particular, $E_T(\phi)$ is the largest T−divisible subspace for the operator T. The study connections between the $E_T(\cdot)$ and $X_T(\cdot)$ subspaces derives from automatic continuity theory. The maximal algebraic spectral subspaces $E_T(\cdot)$ are defined in algebraic terms and the local analytic spectral subspaces $X_T(\cdot)$ are analytically defined. Their structure is much more readily accessible to analysis, see [5], [10],[11], [12], [14], [15], [18] for details.

In [8], B. Johnson and Sinclair were the first to investigate the relationship between automatic continuity question and the existence of divisible subspaces, i.e., for $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, if T is not algebraic and S has a non-trivial S-divisible subspace, then there is a discontinuous linear operators $A : X \to Y$ satisfying $AT = SA$.

In order to have more information about the splitting spaces Rho and Yoo [16] have introduced the following concept of decomposability. An operator $T \in \mathcal{L}(X)$ is called *(E)-super-decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$, there exists an $R \in \mathcal{L}(X)$ commuting with T such that $R^2 = R, \sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I-R)(X)) \subseteq V$. The class of (E)-super-decomposable operators contains all spectral operators, all normal operators on Hilbert spaces, all operators with a totally disconnected spectrum. For various examples and characterizations of (E)-super-decomposable operators, see [16].

2. Maximal spectral spaces of a multiplication operators

Given a commutative complex Banach algebra A, let $\Delta(A)$ stands for the *character space* of A, i.e., the set of all non-trivial multiplicative linear functionals on A. For each $a \in A$, let $\widehat{a} : \Delta(A) \to \mathbb{C}$ denote the corresponding Gelfand transform given by $\widehat{a}(\phi) := \phi(a)$ for all $\phi \in \Delta(A)$. A commutative Banach algebra A is said to be *regular* if for every closed subset $K \subset \Delta(A)$ and any $\phi_0 \notin K$, there exists $x \in A$ such that $\widehat{x}(\phi_0) \neq 0$ and $\widehat{x}(\phi) = 0$ for all $\phi \in K$. Recall that if A is *semi-simple* then $a \to \widehat{a}$ is injective, equivalently, the radical rad(A) of A is $\{0\}$. On $\Delta(A)$ we shall have to consider the *hull-kernel topology*, which is determined by the Kuratouski closure operation

$$cl(B) = \{\phi \in \Delta(A) : \phi(u) = 0 \ \forall\ u \in A \text{ with } \psi(u) = 0 \text{ for each } \psi \in B\}$$

for $B \subseteq \Delta(A)$. In fact, $cl(B) = hul(Ker(B))$. The hull-kernel topology is coarser than the Gelfand topology on $\Delta(A)$ and they coincide if and only if the algebra A is regular, see [4]. For $a \in A$ the multiplication operator $T_a \in \mathcal{L}(A)$ is defined by $T_a x = ax$ for all $x \in A$.

THEOREM 1. *Let $T \in \mathcal{L}(X)$ be (E)-super-decomposable. If T has no non-trivial divisible subspaces, then for each closed $F \subseteq \mathbb{C}$ there* *exists an idempotent operator $R \in \mathcal{L}(X)$ such that the ranges of the R coincides with $X_T(F) = E_T(F)$.*

Proof. Let U be an open neighborhood of F. Then there exists an idempotent $R \in \mathcal{L}(X)$ such that $TR = RT$, $\sigma(T|R(X)) \subseteq U$ and $\sigma(T|(I-R)(X)) \subseteq \mathbb{C} \setminus F$. Since $TR = RT$, we have $(T-\lambda)RE_T(F) = RE_T(F)$ for all $\lambda \in \mathbb{C} \setminus F$, which implies $RE_T(F) \subseteq E_T(F)$ by maximality. A similar argument ensures that $(I-R)E_T(F) \subseteq E_T(F)$. We infer from $\sigma(T|(I-R)(X)) \subseteq \mathbb{C}\setminus F$ that $(I-R)E_T(F) \subseteq (I-R)(X) \subseteq E_T(\mathbb{C} \setminus F)$. And so $(I-R)E_T(F) \subseteq E_T(F) \cap E_T(\mathbb{C} \setminus F) = \{0\}$, since T has no non-trivial divisible subspaces. Hence $E_T(F) = RE_T(F) \subseteq R(X)$. On the other hand, it follows from $\sigma(T|R(X)) \subseteq U$ that $R(X) \subseteq X_T(U) \bigcap E_T(U)$. Hence we have

$$\begin{aligned} R(X) &\subseteq \bigcap_{F \subseteq U} (X_T(U) \cap E_T(U)) \\ &= X_T(F) \bigcap E_T(F) \subseteq X_T(F) \subseteq E_T(F) \subseteq R(X), \end{aligned}$$

because $X_T(F) = \bigcap\{X_T(G) : G \text{ open}, F \subset G\}$. Hence $R(X) = E_T(F) = X_T(F)$, which completes the proof.

We denote by $\mathbf{C}^\infty(\mathbb{C})$ the Frécht algebra of all infinitely differentiable complex valued functions defined on the complex plane $\mathbb{C}$ with the topology of uniform convergence of every derivative on each compact subset of $\mathbb{C}$. An operator $T \in \mathcal{L}(X)$ is called a ***generalized scalar operator*** if there exists a continuous algebra homomorphism $\Phi : \mathbf{C}^\infty(\mathbb{C}) \to \mathcal{L}(X)$ satisfying $\Phi(1) = I$ and $\Phi(z) = T$, where I is the identity operator on X and z is the identity function on $\mathbb{C}$. For a given generalized scalar operator $T \in \mathcal{L}(X)$ and a closed subset F of $\mathbb{C}$, P. Vrbová proved [19] that the existence of a natural number $n \in \mathbb{N}$ such that $X_T(F) = \bigcap_{\lambda \notin F} (T-\lambda)^n X$. From this equality, we have

$$E_T(F) \subseteq \bigcap_{\lambda \notin F,\ m \in \mathbb{N}} (T-\lambda)^m X \subseteq \bigcap_{\lambda \notin F} (T-\lambda)^n X = X_T(F).$$

Hence $X_T(F) = E_T(F)$ for a closed $F \subseteq \mathbb{C}$, and so every generalized scalar operators do not have non-trivial divisible subspaces, since

$E_T(\phi) = X_T(\phi) = \{0\}$. On the other hand, the Volterra operator T defined on $C([0,1])$, the set of all continuous functions with the supremum norm, given by $(Tx)(t) := \int_0^t x(t)dt$ for all $x \in C([0,1])$, and $t \in [0,1]$. Then $\{f \in \mathbf{C}^\infty([0,1]) : f^{(n)}(0) = 0 \text{ for all } n \in \mathbb{N}\}$ is a non-trivial T-divisible subspace and T is quasi-nilpotent.

COROLLARY 2. *Let $T \in \mathcal{L}(H)$ be a normal operator in Hilbert space H and let $\mathcal{E}(\cdot)$ be its spectral measure. Then for each closed $F \subseteq \mathbb{C}$, there exists an idempotent operator $R \in \mathcal{L}(H)$ such that*

$$R(H) = H_T(F) = E_T(F) = \mathcal{E}(F)H = \bigcap_{\lambda \notin F} (T - \lambda)H.$$

Proof. If $Z \subseteq H$ is a T-divisible subspace then by Lemma 5.1 of [18],

$$Z \subseteq \bigcap_{\lambda \in \mathbb{C}} (T - \lambda I)Z \subseteq \bigcap_{\lambda \in \mathbb{C}} (T - \lambda I)H \subseteq \bigcap_{\lambda \in \sigma(T)} (T - \lambda I)H = \{0\},$$

which implies $E_T(\phi) = \{0\}$. By Theorem 1, there exists an idempotent operator $R \in \mathcal{L}(H)$ such that $R(H) = H_T(F) = E_T(F)$, since every normal operator is (E)-super-decomposable operator. By Theorem of [13] and Theorem 5.2 of [18], we have

$$H_T(F) = \mathcal{E}(F)H = \bigcap_{\lambda \notin F} (T - \lambda)H.$$

This completes the proof.

Now we can prove the main result of this paper.

THEOREM 3. *Let A be a semi-simple commutative complex Banach algebra with or without identity, and let $a \in A$ such that the multiplication operator T_a on A has the weak 2-spectral decomposition property. Then for any closed subset F of $\mathbb{C}$, there exists an idempotent element $r \in A$ such that*

$$T_r(A) = \bigcap_{\lambda \notin F} (T_a - \lambda I)A.$$

Moreover, $T_r(A) = A_{T_a}(F) = E_{T_a}(F) = \{x \in A : supp\widehat{x} \subseteq \widehat{a}^{-1}(F)\}$ *is an ideal of* A*, where* $supp\ \widehat{x}$ *may be taken as the hull-kernel closure of the set* $\{\phi \in \triangle(A)\ : \widehat{x}(\phi) \neq 0\}$ *in* $\triangle(A)$.

Proof. Assume that T_a has the weak 2-spectral decomposition property. At first, we claim that the Gelfand transform $\widehat{a}$ is hull-kernel continuous on $\triangle(A)$. Suppose that $\widehat{a}$ is not continuous for the hull-kernel topology of $\triangle(A)$. Then there exists closed $W \subseteq \mathbb{C}$ such that $\widehat{a}^{-1}(W)$ is not a hull in $\triangle(A)$. Let $\phi \in h(k(\widehat{a}^{-1}(W))) \setminus \widehat{a}^{-1}(W)$ and let $\lambda := \phi(a) \notin \widehat{a}^{-1}(W)$. Since T_a has the weak 2-spectral decomposition property, there exists $Y, Z \in Lat(T_a)$ such that

$$A = \overline{Y + Z}, \sigma(T_a|Y) \subseteq \mathbb{C} \setminus \{\lambda\} \quad \text{and} \quad \sigma(T_a|Z) \subseteq \mathbb{C} \setminus \widehat{a}^{-1}(W).$$

For each $y \in Y$ there exists $u \in Y$ such that $y = (T_a - \lambda)u$. Thus $\phi(y) = (\phi(a) - \lambda)\phi(u) = 0$ and hence $\phi = 0$ on Y. Let $\psi \in \widehat{a}^{-1}(W)$ and $\mu := \psi(a) \in W$. For each $z \in Z$ there exists $v \in Z$ such that $(T_a - \mu)v = z$. Thus $\psi(z) = (\psi(a) - \mu)\psi(v) = 0$ and so $\psi = 0$ on Z. It follows from $\phi \in h(k(\widehat{a}^{-1}(W)))$ that $\phi = 0$ on Z. Thus $\phi = 0$ on $A = \overline{Y + Z}$. Since $\phi \in \triangle(A)$, this contradicts our assumption that $\widehat{a}$ is hull-kernel continuous on $\triangle(A)$. Let $A_1 := A \oplus \mathbb{C}$ denote the unitization of a given commutative complex Banach algebra A without identity. It is clear that A_1 is semi-simple and $\widehat{a}$ is also continuous on $\triangle(A_1) = \triangle(A)\bigcup\{\phi_\infty\}$, where each $\phi \in \triangle(A)$ is identified with its canonical extension to A_1 and $\phi_\infty(a + \lambda 1) := \lambda$ for all $a \in A$ and $\lambda \in \mathbb{C}$. Without loss of generality, we may assume that A is semi-simple, commutative Banach algebra with identity. Given an arbitrary open cover $\{U_1, U_2\}$ of $\mathbb{C}$, choose a pair of open sets $W_1, W_2 \subseteq \mathbb{C}$ such that

$$\mathbb{C} \setminus U_1 \subseteq W_1 \subseteq \overline{W_1} \subseteq W_2 \subseteq \overline{W_2} \subseteq U_2.$$

Then both $\widehat{a}^{-1}(\overline{W_1})$ and $\widehat{a}^{-1}(\mathbb{C} \setminus W_2)$ are compact with respect to the hull-kernel topology of $\triangle(A)$. By Corollary 3. 6. 10 of [17], there exists $r \in A$ such that

$$\widehat{r} = 0 \ \text{ on } \widehat{a}^{-1}(\overline{W_1}) \ \text{ and } \ \widehat{r} = 1 \ \text{ on } \widehat{a}^{-1}(\mathbb{C} \setminus W_2).$$

Since $\widehat{r^2} = \widehat{r}$, we obtain $r^2 = r$, by the semi-simplity of A. Let $T_r \in \mathcal{L}(A)$ be given by $T_r x := rx$ for all $x \in A$. It is easily checked that $T_aT_r = T_rT_a$ and $T_r^2 = T_r$. Let $\lambda \in \mathbb{C} \setminus U_1$. Then clearly

$$|(\widehat{a} - \lambda 1)(\phi)| > 0 \quad \text{for all} \quad \phi \in \widehat{a}^{-1}(\mathbb{C} \setminus W_1).$$

Apply Theorem 3. 6. 15 of [17] to obtain $c \in A$ such that $(\widehat{a} - \lambda 1)\widehat{cr} = \widehat{r}$ on $\Delta(A)$, which implies $(a - \lambda 1)cr = r$, i.e., $(T_a - \lambda I)ST_r = T_r$ where $Sx := cx$ for all $x \in A$. Hence $(T_a - \lambda I)|T_r(A)$ is invertible and so $\sigma(T_a|T_r(A)) \subseteq U_1$. Proceeding in the same way, we obtain $\sigma(T_a|(I - T_r)(A)) \subseteq U_2$, and hence T_a is (E)-super-decoposable. If $Y \subseteq A$ is T_a-divisible subspace, then we have

$$Y = \bigcap_{\lambda \in \mathbb{C}} (T_a - \lambda I)Y \subseteq \bigcap_{\lambda \in \mathbb{C}} (T_a - \lambda I)A \subseteq rad\ (A),$$

where $rad\ (A)$ denotes the radical of A. The semi-simplicity of A implies that

$$Y = \bigcap_{\lambda \in \mathbb{C}} (T_a - \lambda I)A = \{0\},$$

and so $E_T(\phi) = \{0\}$. Hence by Theorem 1, $T_r(A) = E_{T_a}(F) = A_{T_a}(F)$. Since the spectral maximal space $A_{T_a}(F) = T_r(A)$ is hyperinvariant, $A_{T_a}(F)$ is an ideal of A. On the other hand, it follows from Theorem 6. 2. 5 of [5] that

$$A_{T_a}(F) = \{x \in A : supp\widehat{x} \subseteq \widehat{a}^{-1}(F)\}.$$

Finally, we claim that

$$A_{T_a}(F) = \bigcap_{\lambda \notin F} (T_a - \lambda I)A.$$

The inclusion $A_{T_a}(F) \subseteq \bigcap_{\lambda \in \mathbb{C} \setminus F} (T_a - \lambda I)A$ is clear from the elementary fact that $(T_a - \lambda)A_{T_a}(F) = A_{T_a}(F)$ for all $\lambda \notin F$. To prove the reverse inclusion, take any open neighborhood V of F. Then $\{V, \mathbb{C} \setminus F\}$ is an open covering of the complex plane $\mathbb{C}$. As T_a is (E)-super-decomposable, there exists an idempotent $R \in \mathcal{L}(X)$ commuting with T_a such that

$$\sigma(T_a|R(A)) \subseteq V \quad \text{and} \quad \sigma(T_a|(I - R)(A)) \subseteq \mathbb{C} \setminus F.$$

Let

$$Z := \bigcap_{\lambda \notin F} (T_a - \lambda I)A.$$

It is easily checked that

$$(I - R)(Z) \subseteq \bigcap_{\lambda \in \mathbb{C}} (T_a - \lambda I)A \subseteq rad(A),$$

which implies that $Z \subseteq R(A) \subseteq A_{T_a}(V)$. Since $X_T(\cdot)$ is known to preserve intersection, $Z \subseteq \bigcap_{F \subseteq V} A_{T_a}(V) = A_{T_a}(F)$. Hence we have

$$A_{T_a}(F) = \bigcap_{\lambda \notin F} (T_a - \lambda I)A.$$

This completes the proof.

Note that the preceding proof shows that actually T_a is (E)-super-decompsable if and only if T_a is super-decompsable if and only if T_a is decompsable if and only if T_a has the weak 2-spectral decomposition property if and only if the Gelfand transform $\widehat{a}$ is hull-kernel continuous on $\triangle(A)$.

THEOREM 4. *Let A be a commutative complex Banach algebra with or without identity, and let $a \in A$ such that the Gelfand transform $\widehat{a}$ is hull-kernel continuous on $\triangle(A)$. Then for any closed $F \subseteq \mathbb{C}$, there exists an idempotent element $[r] \in B := A/rad\ (A)$, the coset of $r \in A$, such that*

$$T_{[r]}(B) = B_{T_{[a]}}(F) = E_{T_{[a]}}(F) = \bigcap_{\lambda \notin F} (T_{[a]} - \lambda I)B$$
$$= \{\ [x] \in B\ :\ supp\,\widehat{[x]} \subseteq \widehat{[a]}^{-1}(F)\},$$

where $rad\ (A)$ denotes the radical of A.

Proof. It is clear that $B := A/rad(A)$ is a semi-simple commutative Banach algebra. Let $\pi : A \longrightarrow A/rad\ (A)$ be the quotient map. Then by Theorem 23.5 of [4], the map

$$\pi^* : \triangle(A/rad\ (A)) \longrightarrow \triangle(A)$$

defined by $\pi^*(\phi) = \phi \circ \pi$ for $\phi \in \Delta(A/rad\,(A))$ is a homeomorphism. Hence $\widehat{\pi(a)} = \widehat{[a]}$ is hull-kernel continuous on $\Delta(A/rad\,(A))$, since $\widehat{a}$ is hull-kernel contiuous on $\Delta(A)$. It is easily seen that

$$T_{[a]} : A/rad(A) \longrightarrow A/rad(A),$$

given by $T_{[a]}[x] = [ax]$ for all $[x] \in A/rad(A)$, is (E)-super-decomposable. By Theorem 3, one has the results. This completes the proof.

An obvious combination of Theorem 3 and Theorem 2 of [6] leads to the following results.

COROLLARY 5. *Let A be a semi-simple, regular commutative complex Banach algebra with or without identity, and let $a \in A$. Then for any closed subset F of $\mathbb{C}$, there exists an idempotent element $r \in A$ such that*

$$T_r(A) = A_{T_a}(F) = E_{T_a}(F) = \bigcap_{\lambda \notin F} (T_a - \lambda I)A.$$

The following Corollary applies, for instance, to any commutative C^*-algebra, since every commutative C^*-algebra is semi-simple and regular Banach algebra. We have the following.

COROLLARY 6. *Let A be a commutative C^*-algebra, and let $a \in A$. Then for any closed $F \subseteq \mathbb{C}$, there exists an idempotent element $r \in A$ such that*

$$T_r(A) = A_{T_a}(F) = E_{T_a}(F) = \bigcap_{\lambda \notin F} (T_a - \lambda I)A.$$

Proof. Every commutative C^*-algebra is regular and the hull-kernel topology coincides with the Gelfand topology on $\Delta(A)$. Thus T_a has the weak 2-spectral decomposition property. By Theorem 3, one has the results.

THEOREM 7. *Let X be a complex Banach space, and let A denote a semi-simple commutative Banach algebra with identity. Assume that $\Phi : A \longrightarrow \mathcal{L}(X)$ is an algebra homomorphism. If $T_a \in \mathcal{L}(A)$ has* *the weak 2-spectral decomposition property, then the corresponding operator $T := \Phi(a) \in \mathcal{L}(X)$ is (E)-super-decomposable.*

Proof. It is easily checked that the Gelfand transform $\widehat{a}$ is hull-kernel continuous on $\triangle(A)$. For every open covering $\{U, V\}$ of $\mathbb{C}$, choose a pair of open sets $G,\ K \subseteq \mathbb{C}$ such that $\overline{G} \subseteq U, \overline{K} \subseteq V$ and $G \cup K = \mathbb{C}$. Then both $\widehat{a}^{-1}(\mathbb{C} \setminus G)$ and $\widehat{a}^{-1}(\mathbb{C} \setminus K)$ are disjoint, compact with respect to the Gelfand topology of the compact space $\triangle(A)$. By Corollary 3. 6.10 of [17], we deduce that there exists $b \in A$ such that

$$\widehat{b} = 0 \ \text{ on } \ \widehat{a}^{-1}(\mathbb{C} \setminus G) \ \text{ and } \ \widehat{b} = 1 \ \text{ on } \ \widehat{a}^{-1}(\mathbb{C} \setminus K).$$

Let $R := \Phi(b) \in \mathcal{L}(A)$. Clearly, T commutes with R. From $\widehat{r^2} = \widehat{r}$ on $\triangle(A)$, we know that $r^2 = r$ by semi-simplicity of A and so $R^2 = R$. We claim that $\sigma(T|R(X)) \subseteq U$. Let $\lambda \in \mathbb{C} \setminus U$. Then $|(\widehat{a} - \lambda 1)| > 0$ holds on the hull $\widehat{a}^{-1}(G)$. Thus we may apply Theorem 3. 6. 15 of [17] to obtain $d \in A$ such that $(\widehat{a} - \lambda 1)\widehat{d} = 1$ on $\widehat{a}^{-1}(\overline{G})$. Since $\widehat{b} = 0$ on $\widehat{a}^{-1}(\mathbb{C} \setminus G)$, we have $(\widehat{a} - \lambda)\widehat{b}\widehat{d} = \widehat{b}$ on $\triangle(A)$, which implies that $(a - \lambda)bd = b$. Let $S := \Phi(d) \in \mathcal{L}(X)$. Apply the homomorphism Φ to the equation $(a - \lambda 1)bd = b$, we have $(T - \lambda I)SR = S(T - \lambda I)R = R$. Thus $(T - \lambda I)S = S(T - \lambda I) = I$ on $R(X)$, the range of R. Hence $(T - \lambda I)|R(X)$ is invertible, i.e., $\lambda \in \rho(T|R(X))$, and so $\sigma(T|R(X)) \subseteq U$. A similar argument can be used to show that $\sigma(T|(I - R)(X)) \subseteq V$. Hence T is (E)-super-decomposable. This completes the proof.

COROLLARY 8. *In the situation of Theorem 7, if $\bigcap_{\lambda \in \mathbb{C}}(T - \lambda)X = \{0\}$, then for any closed $F \subseteq \mathbb{C}$, there exists an idempotent operator $R \in \mathcal{L}(X)$ such that*

$$R(X) = X_T(F) = E_T(F) = \bigcap_{\lambda \notin F}(T - \lambda)X.$$

Note that for a given commutative, semi-simple and regular Banach algebra A over $\mathbb{C}$, the left regular representation $\Phi : A \longrightarrow \mathcal{L}(A)$ given by $\Phi(a) := T_a$ for all $a \in A$. It follows from [16] that T_a is (E)-super-decomposable. Moreover, every multiplication operator T_a on A satisfies

$$\bigcap_{\lambda \in \mathbb{C}}(T_a - \lambda)A \subseteq rad(A),$$

the radical of A, so that $\bigcap_{\lambda\in\mathbb{C}}(T_a-\lambda)A=\{0\}$, since A is semi-simple. It follows from Corollary 8 that for each closed $F\subseteq\mathbb{C}$ there exists an idempotent operator $R\in\mathcal{L}(A)$ such that

$$R(A)=A_{T_a}(F)=E_{T_a}(F)=\bigcap_{\lambda\notin F}(T_a-\lambda)A.$$

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Jae Chul Rho
Department of Mathematices
Sogang University
Seoul 121-742, Korea

Jong-Kwang Yoo
Department of Mathematices
Chodang University
Muan 534-800, Korea