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ON A CLASS OF TERNARY COMPOSITION ALGEBRAS

ALBERTO ELDUQUE

1. Introduction

When dealing with a Lie group or, in general, with an analytic loop or quasigroup, its symmetry is broken by the election of the distinguished identity element.

In [**N**], a new multiplication is defined on a loop by means of $x \cdot_a y = x(a\backslash y)$, where $a\backslash y$ is the left division of y by a. With this new multiplication, the element a turns out to be the new identity element. More generally, in studying quasigroups, the existence of a Malcev operation (see [**Sm**]) plays an important role. This is a ternary operation such that $\theta(x,x,y) = \theta(y,x,x) = y$ for any x, y and can be defined as $\theta(x,y,z) = (x/(y\backslash y))(y\backslash z)$, which for loops (or even just left loops) becomes $\theta(x,y,z) = x(y\backslash z) = x \cdot_y z$.

As a privileged example, the seven-dimensional sphere $S^7 = \{x \in \mathbb{R}^8 : \langle x|x\rangle = 1\}$, where $\langle\cdot|\cdot\rangle$ is the standard inner product, has a structure of analytic Moufang loop (but not a Lie group) inherited from the multiplication in the real division octonion algebra $\mathbb{O}$. Hence, if we identify S^7 with $\{x \in \mathbb{O} : n(x) = 1\}$, where n is the norm of $\mathbb{O}$, then the product in $\mathbb{O}$ of two elements in S^7 belongs to S^7. Although S^7 is perfectly homogeneous, this identification points out the distinguished identity element of $\mathbb{O}$ and breaks the symmetry. The corresponding Malcev operation here is given by

$$\theta(x,y,z) = x(y\backslash z) = x(y^{-1}z) = x(\bar{y}z),$$

where $y \mapsto \bar{y}$ is the conjugation in $\mathbb{O}$.

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This suggests the investigation of the triple product $\{x,y,z\} = x(\bar{y}z)$ or, alternatively, $\{x,y,z\} = (x\bar{y})z$, defined on any Cayley Dickson algebra. This is the aim of the present paper.

These triple products have been investigated over $\mathbb{R}$ by Shaw [**Sh1–3**], they are related to the so called vector cross products, which have been studied in [**Ec**] and [**B–G**] and have recently been used by Okubo [**O**] to find solutions to the Yang–Baxter equation. Both Shaw and Okubo put emphasis in the fact that these triple products have larger groups of symmetries (automorphisms) than the corresponding Cayley–Dickson algebras. Shaw determines these groups in [**Sh3**]. Many of our results will be a general reformulation of results in [**Sh1–3**].

The approach used by Shaw is to determine the properties of the triple products without appealing to the underlying (binary) composition algebra. However, in dealing with the automorphism group, he breaks the symmetry by fixing one element. This is the same as breaking the symmetry by considering an associated composition algebra.

The approach here will be to consider these triple products over arbitrary fields of characteristic not two from the opposite point of view. From any composition algebra of dimension 4 or 8, the properties of the associated triple products will be deduced from properties of the composition algebras. In particular, the automorphism groups of these systems in dimension 8 appear very naturally as Spin groups and the automorphism group of the Cayley Dickson algebra will be recovered as the isotropy subgroup at the identity element, thus providing a description of the automorphism group of any Cayley Dickson algebra by means of multiplications by elements of trace zero. This action of a rank 7 spin group on a Cayley–Dickson algebra was considered also in [**E–M**] to study the reductive homogeneous space $S^7 = Spin(7)/G_2$. In that paper, the fact that the automorphism group of the Cayley–Dickson algebra coincides with the isotropy subgroup of $Spin(7)$ was proved only for the real division octonion algebra, by using that both groups are connected Lie groups with the same Lie algebra. The use of the triple products here allows to get this result very easily in complete generality.

Schwarz considered in [**Schw**] the invariants of G_2 and $Spin(7)$ over $\mathbb{C}$. He obtained that $Spin(7)$ is precisely the set of linear maps in $GL(C)$, with C the Cayley–Dickson algebra over $\mathbb{C}$, preserving the norm of C and a specific skew–symmetric multilinear map $C^4 \to \mathbb{C}$. This mapping appears naturally in our description.

It must be remarked here that all the triple composition algebras have been determined, up to isotopy, by McCrimmon [**McC**].

The paper is structured as follows, Section 2 will introduce the class of ternary composition products studied here and will relate them to composition algebras. Section 3 will be devoted to the calculation of the automorphism groups and Lie algebras of derivations and to derive some consequences of them. Finally, in Section 4, a multilinear skew–symmetric form which appears in previous sections will be expressed in terms of traces of products of elements in Cayley–Dickson algebras. This will be related to the work on invariants by Schwarz [**Schw**].

Throughout the paper, all vector spaces and algebras will be considered over a field F of characteristic not two. For standard facts about composition algebras, see [**Sch**] and [**ZSSS**].

2. Three–fold vector cross products and related triple composition products

In this section, the definitions and some relevant properties of the triple products considered will be given. These latter are classified in two types in [**Sh2**] in case the dimension is 8 (over $\mathbb{R}$). The same classification will be given here in general but in a different way, and a property which separates these two types, which appear in [**Sh2**; Theorem 7.5] as a byproduct, will be given here a central role. This is the property used by Okubo [**O**] in defining what he calls the quaternionic and octonionic triple systems, which he uses to obtain solutions of the Yang–Baxter equation.

A *three–fold vector cross product* on a vector space V equipped with a nondegenerate symmetric bilinear form $\langle\cdot|\cdot\rangle$ is a trilinear map

$$\begin{aligned} X : V \times V \times V &\longrightarrow V \\ (a,b,c) &\mapsto X(a,b,c) \end{aligned}$$

satisfying

$$\langle X(a_1,a_2,a_3)|a_i\rangle = 0 \text{ for any } i = 1,2,3 \tag{1}$$

$$\langle X(a_1,a_2,a_3)|X(a_1,a_2,a_3)\rangle = \det(\langle a_i|a_j\rangle) \tag{2}$$

(see [**B–G**]). Linearizing (1) we get

$$\langle X(a_1, a_1, a_3)|a_2\rangle = -\langle X(a_1, a_2, a_3)|a_1\rangle = 0,$$

so $X(a, a, b) = 0$ for any $a, b \in V$. Similarly, $X(a, b, b) = 0$, so that X is skew-symmetric. Therefore, X can be viewed as a linear map $\Lambda^3 V \to V$. If $\dim V \leq 2$, then the only possibility is $X = 0$. Also, notice that $\langle X(a, b, c)|d\rangle$ is skew–symmetric in its arguments.

Now, assume that the vector space with a nondegenerate symmetric product $(V, \langle\cdot|\cdot\rangle)$ is equipped with a triple product

$$\begin{aligned} \{\cdot,\cdot,\cdot\} : V \times V \times V &\longrightarrow V \\ (a, b, c) &\mapsto \{a, b, c\} \end{aligned}$$

satisfying

i) $\{a, a, b\} = \langle a|a\rangle b = \{b, a, a\}$,

ii) $\langle\{a, b, c\}|\{a, b, c\}\rangle = \langle a|a\rangle\langle b|b\rangle\langle c|c\rangle$,

for any $a, b, c \in V$. Then, following [**Sh1**], $\{\cdot,\cdot,\cdot\}$ will be called a *3C–product* on $(V, \langle\cdot|\cdot\rangle)$. The triple $(V, \langle\cdot|\cdot\rangle, \{\cdot,\cdot,\ \})$ will be called a *3C–algebra*.

These two definitions are equivalent:

Proposition 1 ([**Sh1**]). *Let V be a vector space equipped with a nondegenerate symmetric bilinear form $\langle\cdot|\cdot\rangle$ as above. If X is a three-fold vector cross product on V, then the triple product $\{\cdot,\cdot,\cdot\}$ defined by*

$$\{a, b, c\} = X(a, b, c) + \langle a|b\rangle c + \langle b|c\rangle a - \langle a|c\rangle b \tag{3}$$

is a 3C–product on V. Conversely, if $\{\cdot,\cdot,\cdot\}$ is a 3C–product on $(V, \langle\cdot|\cdot\rangle)$, then the trilinear product X defined by (3) is a three–fold vector cross product.

We include a proof for completeness:

Proof. If X is a three–fold vector cross product and $\{\cdot,\cdot,\cdot\}$ is defined by (3), then $\{a,a,c\} = \langle a|a\rangle c = \{c,a,a\}$ for any $a, c \in V$ and

$$\begin{aligned}
\langle\{a,b,c\}|\{a,b,c\}\rangle &= \langle X(a,b,c)|X(a,b,c)\rangle + \\
&\quad \langle\langle a|b\rangle c + \langle b|c\rangle a - \langle a|c\rangle b \,|\, \langle a|b\rangle c + \langle b|c\rangle a - \langle a|c\rangle b\rangle \\
&= \det\begin{pmatrix} \langle a|a\rangle & \langle a|b\rangle & \langle a|c\rangle \\ \langle a|b\rangle & \langle b|b\rangle & \langle b|c\rangle \\ \langle a|c\rangle & \langle b|c\rangle & \langle c|c\rangle \end{pmatrix} + \\
&\quad \langle\langle a|b\rangle c + \langle b|c\rangle a - \langle a|c\rangle b \,|\, \langle a|b\rangle c + \langle b|c\rangle a - \langle a|c\rangle b\rangle \\
&= \langle a|a\rangle\langle b|b\rangle\langle c|c\rangle \qquad (4)
\end{aligned}$$

Conversely, if $\{\cdot,\cdot,\cdot\}$ is a $3C$–product on V and X is defined by (3), then X is skew–symmetric and

$$\langle c|X(a,b,c)\rangle = \langle c|\{a,b,c\} - \langle a|b\rangle c\rangle = \langle c|\big(L(a,b) - \langle a|b\rangle I\big)(c)\rangle,$$

where $L(a,b) : x \mapsto \{a,b,x\}$ and I is the identity mapping.

But $L(a,a) = \langle a|a\rangle I$, so

$$L(a,b) + L(b,a) = 2\langle a|b\rangle I. \qquad (5)$$

Also, $L(a,b)^*L(a,b) = \langle a|a\rangle\langle b|b\rangle I$, where * denotes the adjunction with respect to $\langle\cdot|\cdot\rangle$, so $L(a,b)^*L(a,c) + L(a,c)^*L(a,b) = 2\langle a|a\rangle\langle b|c\rangle I$, and with $c = a$ this gives

$$L(a,b)^*L(a,a) + L(a,a)^*L(a,b) = 2\langle a|a\rangle\langle a|b\rangle I. \qquad (6)$$

Since $L(a,a) = \langle a|a\rangle I = L(a,a)^*$, we conclude from (5) and (6) that

$$L(a,b)^* = L(b,a) \qquad (7)$$

provided $\langle a|a\rangle \neq 0$ and, by Zariski density (extend scalars if necessary), for any $a, b \in V$. Now, (5) and (7) imply that $L(a,b) - \langle a|b\rangle I$ is skew–symmetric, so that $\langle c|\big(L(a,b) - \langle a|b\rangle I\big)(c)\rangle = 0$, as required.

Finally, the computation in (4) implies that X is a three–fold vector cross product. $\square$

The next theorem appears essentially in [**B–G**; Theorem 5.2], we also enclose the proof, with some simplifications:

THEOREM 2. *Let $\{\cdot,\cdot,\cdot\}$ be a 3C product on a pair $(V,\langle\cdot|\cdot\rangle)$ and let e be any element of V with $\langle e|e\rangle \neq 0$. Define a multiplication in V by*

$$ac = \langle e|e\rangle^{-1}\{a,e,c\}. \tag{8}$$

Then:

i) *With this multiplication, V is a composition algebra with identity element e and associated bilinear form $\langle x|y\rangle_1 = \langle e|e\rangle^{-1}\langle x|y\rangle$.*

ii) *Either $\{a,b,c\} = \langle e|e\rangle(a\bar{b})c$ (type I) or $\{a,b,c\} = \langle e|e\rangle a(\bar{b}c)$ (type II), for any $a,b,c \in V$, where $x \mapsto \bar{x}$ is the standard involution in the composition algebra V.*

Conversely, given any composition algebra V with symmetric bilinear form $\langle\cdot|\cdot\rangle_1$ and identity element e, and any nonzero scalar $\alpha \in F$, either of the two possibilities $\{a,b,c\} = \alpha(a\bar{b})c$ or $\alpha a(\bar{b}c)$ define a 3C-product on $(V,\langle\cdot|\cdot\rangle)$, with $\langle\cdot|\cdot\rangle = \alpha\langle\cdot|\cdot\rangle_1$, from which we recover the composition product by means of (8).

Proof. i) and the converse are clear. For ii), if $a = e$, $\{e,b,c\} = -\{b,e,c\} + 2\langle e|b\rangle c = -\langle e|e\rangle bc + 2\langle e|e\rangle\langle e|b\rangle_1 ec = \langle e|e\rangle\bar{b}c$. The same applies for $c = e$, and if $b = e$, $\{a,e,c\} = \langle e|e\rangle ec$ by the definition. If $\dim V \leq 2$ we are finished. Also, if $a = b$, $\{a,a,c\} = \langle a|a\rangle c = \langle e|e\rangle\langle a|a\rangle_1 c = \langle e|e\rangle a(\bar{a}c) = \langle e|e\rangle(a\bar{a})c$. The same applies if $b = c$. If $a = c$, $\{a,b,a\} = -\{b,a,a\} + 2\langle a|b\rangle a = -\langle e|e\rangle\langle a|a\rangle_1 b + 2\langle e|e\rangle\langle a|b\rangle_1 a = \langle e|e\rangle(a\bar{b})a = \langle e|e\rangle a(\bar{b}a)$.

Therefore, it is enough by linearity to check ii) with a,b,c different members of an orthogonal basis of $(Fe)^\perp$. We are left with two cases:

a) $Fe + Fa + Fb + Fc = H$ is a quaternion subalgebra of V. Then we may assume that $ab = -ba = c$ so

$$(a\bar{b})c = a(\bar{b}c) = -a(\bar{b}(ba)) = \langle b|b\rangle_1\langle a|a\rangle_1 e,$$

and $\{a,b,c\} = X(a,b,c)$ is orthogonal to a,b and c, where X is as in the previous proposition. Thus, $\{a,b,c\} = \beta e + d$, with $d \in H^\perp$ and

$$\begin{aligned}\beta = \langle\beta e|e\rangle_1 &= \langle X(a,b,c)|e\rangle_1 = \langle X(a,e,b)|c\rangle_1\\ &= \langle e|e\rangle\langle ab|c\rangle_1 = \langle e|e\rangle\langle ab|ab\rangle_1 = \langle e|e\rangle\langle a|a\rangle_1\langle b|b\rangle_1\end{aligned}$$

Therefore, $\{a,b,c\} = \langle e|e\rangle(a\bar{b})c = \langle e|e\rangle a(\bar{b}c)$ if $V = H$ ($\dim V = 4$) or $d = 0$. But, if $\dim V = 8$, linearizing (2) with respect to a_1 and then with respect to a_2 gives

$$\langle X(a,b,c)|X(xc,e,c)\rangle + \langle X(xc,b,c)|X(a,e,c)\rangle = 0$$

for any $x \in H^\perp$, and since $X(a,e,c) = \langle e|e\rangle ac = \gamma b$ for some γ, we obtain $\langle X(a,b,c)|(xc)c\rangle = 0$, so $\langle X(a,b,c)|x\rangle = 0$, $\langle X(a,b,c)|H^\perp\rangle = 0$, $d = 0$ and $X(a,b,c) = \beta e$.

b) $\dim V = 8$ and c is orthogonal to the quaternion subalgebra $H = Fe + Fa + Fb + Fab$. Then $V = H + Hc$ and $(a\bar{b})c = -(ab)c = a(bc) = -a(\bar{b}c)$. Now, $\langle X(a,b,c)|e\rangle = -\langle X(a,b,e)|c\rangle = \langle ab|c\rangle = 0$. By item a), $\langle X(a,b,c)|ab\rangle = -\langle X(a,b,ab)|c\rangle = 0$ and similarly, considering the quaternion subalgebras $H' = Fe + Fa + Fc + Fac$ and $H'' = Fe + Fb + Fc + Fbc$, we get by item a) that $0 = \langle X(a,b,c)|ac\rangle = \langle X(a,b,c)|bc\rangle$. Since $\{e,a,b,ab,c,ac,bc,(ab)c\}$ is an orthogonal basis of V, it follows that $X(a,b,c) = \{a,b,c\} = -\epsilon\langle e|e\rangle(ab)c = \epsilon\langle e|e\rangle(a\bar{b})c$ for some $\epsilon \in F$. Taking norms we get that $\epsilon = \pm 1$.

Therefore we conclude that for any a,b,c in V, either $\{a,b,c\} = \langle e|e\rangle(a\bar{b})c$ or $\{a,b,c\} = \langle e|e\rangle a(\bar{b}c)$ (or both). By Zariski topology (extend scalars if necessary to get an infinite field), the sets $\{(a,b,c) \in V^3 : \{a,b,c\} \neq \langle e|e\rangle(a\bar{b})c\}$ and $\{(a,b,c) \in V^3 : \{a,b,c\} \neq \langle e|e\rangle a(\bar{b}c)\}$ are open, so if they are nonempty, they intersect nontrivially, a contradiction that proves ii) $\square$

REMARK. If $\{\cdot,\cdot,\cdot\}$ is a $3C$–product on a pair $(V, \langle\cdot|\cdot\rangle)$ and e and f are two elements of V with $\langle e|e\rangle \neq 0 \neq \langle f|f\rangle$, then the composition algebras constructed in Theorem 2 by means of e and f are isomorphic. To prove this it is enough to check that the corresponding norm forms are equivalent. But if $\varphi : V \longrightarrow V$ is the linear map given by $\varphi(x) = \langle e|e\rangle^{-1}\{e,x,f\}$, then:

$$\langle f|f\rangle^{-1}\langle\varphi(x)|\varphi(x)\rangle = \langle f|f\rangle^{-1}\langle e|e\rangle^{-2}\langle\{e,x,f\}|\{e,x,f\}\rangle = \langle e|e\rangle^{-1}\langle x|x\rangle,$$

which shows that they are indeed equivalent.

The distinction between types I and II in the Theorem above can be done easily by means of the following result (see also [**Sh2**; Theorem 7.5]):

PROPOSITION 3. *Let X be a three-fold vector cross product on $(V,\langle\cdot|\cdot\rangle)$, let $\{\cdot,\cdot,\cdot\}$ be the associated 3C-product and let $\Phi : V^4 \to F$ be the skew-symmetric multilinear form given by $\Phi(a,b,c,d) = \langle a|X(b,c,d)\rangle$. Then, for any $a_i, b_j \in V$, $i,j = 1,2,3$:*
(9)

$$\langle X(a_1,a_2,a_3)|X(b_1,b_2,b_3)\rangle = \det(\langle a_i|b_j\rangle) + \epsilon \sum_{\sigma \text{ even}} \sum_{\tau \text{ even}} \langle a_{\sigma(1)}|b_{\tau(1)}\rangle \Phi(a_{\sigma(2)}, a_{\sigma(3)}, b_{\tau(2)}, b_{\tau(3)}),$$

where $\epsilon = 0$ if $\dim V = 1,2$ or 4, $\epsilon = 1$ if $\dim V = 8$ and $\{a,b,c\} = \langle e|e\rangle(a\bar{b})c$ in the Theorem above, and $\epsilon = -1$ if $\dim V = 8$ and $\{a,b,c\} = \langle e|e\rangle a(\bar{b}c)$ in the Theorem above.

REMARK. This means that the two types I and II of the Theorem above can be distinguished from the beginning and do not depend on the particular choice of the element e.

Proof. If $\dim V = 1$ or 2, then $X = \Phi = 0 = \det(\langle a_i|b_j\rangle)$ and we are done. If $\dim V = 4$, we have also that

$$\sum_{\sigma \text{ even}} \sum_{\tau \text{ even}} \langle a_{\sigma(1)}|b_{\tau(1)}\rangle \Phi(a_{\sigma(2)}, a_{\sigma(3)}, b_{\tau(2)}, b_{\tau(3)}) = 0.$$

To prove this, and since this expression is skew-symmetric in the a_i's and in the b_j's, and multilinear, it is enough to assume that the a_i's and b_j's are members of an orthogonal basis $\{e_1,e_2,e_3,e_4\}$ of V. Hence, we may assume $a_1 = e_1 = b_1$, $a_2 = e_2 = b_2$, $a_3 = e_3$ and $b_3 = e_4$. But in this case what we obtain is

$$\langle e_1|e_1\rangle\Phi(e_2,e_3,e_2,e_4) + \langle e_2|e_2\rangle\Phi(e_3,e_1,e_4,e_1) = 0$$

and, by linearizing (2) with respect to a_3:

$$\langle X(e_1,e_2,e_3)|X(e_1,e_2,e_4)\rangle = \det\begin{pmatrix} \langle e_1|e_1\rangle & \langle e_1|e_2\rangle & \langle e_1|e_4\rangle \\ \langle e_2|e_1\rangle & \langle e_2|e_2\rangle & \langle e_2|e_4\rangle \\ \langle e_3|e_1\rangle & \langle e_3|e_2\rangle & \langle e_3|e_4\rangle \end{pmatrix} = 0.$$

Finally, if $\dim V = 8$, $e \in V$ with $\langle e|e\rangle \neq 0$, $ac = \langle e|e\rangle^{-1}\{a,e,c\}$ and $\{a,b,c\} = \langle e|e\rangle(a\bar{b})c$ (the other case is similar), put $\epsilon = 1$ in (9), and it is enough to check (9) with the a_i's and b_j's in an orthogonal basis of V with different a_i's and different b_j's. We may assume that the basis includes a_1, a_2, a_3 and $X(a_1, a_2, a_3)$, which is orthogonal to the others. Three cases are possible:

i) Assume there are at least two elements in common in the families a_1, a_2, a_3 and b_1, b_2, b_3. Then, the second term on the right of (9) is zero and (9) follows from the linearization of (2).

ii) Assume there is only one common element, say $a_3 = b_3$. Then $\det(\langle a_i|b_j\rangle) = 0$ and

$$\begin{aligned}
&\langle X(a_1, a_2, a_3)|X(b_1, b_2, a_3)\rangle \\
&= \langle e|e\rangle^2 \langle (a_1\bar{a}_2)a_3|(b_1\bar{b}_2)a_3\rangle \\
&= \langle e|e\rangle \langle (a_1\bar{a}_2)a_3|(b_1\bar{b}_2)a_3\rangle_1 = \langle e|e\rangle\langle a_3|a_3\rangle_1\langle a_1\bar{a}_2|b_1\bar{b}_2\rangle_1 \\
&= \langle a_3|a_3\rangle\langle (a_1\bar{a}_2)b_2|b_1\rangle_1 \\
&= \langle a_3|a_3\rangle\langle b_1|X(a_1, a_2, b_2\rangle = \langle a_3|a_3\rangle\Phi(a_1, a_2, b_1, b_2),
\end{aligned}$$

so we obtain (9).

iii) Finally, if there is no element in common, the right hand side of (9) is 0 and for the left hand side we define $x * y = \langle a_3|a_3\rangle^{-1}\{x, a_3, y\}$. Then, $H = Fa_3 + Fa_1 + Fa_2 + Fa_1 * a_2$ is a quaternion subalgebra of the Cayley–Dickson algebra $(V, *)$. If any of the b_i's, say b_3 is $\gamma a_1 * a_2$, then

$$\langle X(a_1, a_2, a_3)|X(b_1, b_2, b_3)\rangle = \mu\langle b_3|X(b_1, b_2, b_3)\rangle = 0$$

for some scalar μ. Otherwise, $V = H + H * b_3$, $b_1, b_2, b_3 \in H * b_3 = H^\perp$ and $X(b_1, b_2, b_3) = \{b_1, b_2, b_3\} = \gamma(b_1 * b_2) * b_3$ or $\gamma b_1 * (b_2 * b_3)$ for some $\gamma \in F$, so $X(b_1, b_2, b_3) \in H^\perp$ and also $\langle X(a_1, a_2, a_3)|X(b_1, b_2, b_3)\rangle = 0$. □

We have used that if H is a (generalized) quaternion subalgebra of a Cayley–Dickson algebra and $b \in H^\perp$, then $V = H + Hb$, $H(Hb) + (Hb)H \subseteq Hb$ and $(Hb)^2 \subseteq H$.

REMARK. It is straightforward to prove that given a $3C$–product, the automorphism group is

$$\begin{aligned}
\mathrm{Aut}(V, \{\cdot,\cdot,\cdot\}) &= \{\varphi \in GL(V) : \varphi \text{ is orthogonal for } \langle\cdot|\cdot\rangle \text{ and} \\
&\qquad \varphi \in \mathrm{Aut}(V, X)\} \\
&= \{\varphi \in GL(V) : \varphi \text{ leaves invariantboth } \langle\cdot|\cdot\rangle \text{ and } \Phi\}.
\end{aligned}$$

Also, given $(V_1, \{\cdot,\cdot,\cdot\}_1)$ and $(V_2, \{\cdot,\cdot,\cdot\}_2)$ and $\varphi : V_1 \to V_2$ a linear map, then φ is an isomorphism if and only if φ is an isometry of $(V_1, \langle\cdot|\cdot\rangle_1)$ into $(V_2, \langle\cdot|\cdot\rangle_2)$ and φ is an isomorphism of (V_1, X_1) into (V_2, X_2). In particular, if $\dim V = 8$, no $3C$-product of type I is isomorphic to another of type II. This gives an alternate proof of [**B–G**; Lemma (5.3)].

Also notice that if X is a three-fold vector cross product of type I, then $-X$ is of type II (just look at (9)). Of course, we say that X is of type I if so is the associated $3C$-product.

THEOREM 4. (Isomorphism condition) *Let $\{\cdot,\cdot,\ \}_i$ be a $3C$-product on a pair $(V_i, \langle\cdot|\cdot\rangle_i)$, $i = 1, 2$. Then, the triple products $(V_1, \{\cdot,\cdot,\cdot\}_1)$ and $(V_2, \{\cdot,\cdot,\cdot\}_2)$ are isomorphic if and only if they are of the same type and the bilinear forms $\langle\cdot|\cdot\rangle_i$, $i = 1, 2$, are equivalent.*

Proof. It is clear that if $\varphi : (V_1, \{\cdot,\cdot,\cdot\}_1) \longrightarrow (V_2, \{\cdot,\cdot,\cdot\}_2)$ is an isomorphism, then for any $e, a \in V$,

$$\begin{aligned}\langle e|e\rangle_1 \varphi(a) &= \varphi(\langle e|e\rangle_1 a) = \varphi(\{e, e, a\}_1) \\ &= \{\varphi(e), \varphi(e), \varphi(a)\}_2 = \langle \varphi(e)|\varphi(e)\rangle_2 \varphi(a) ,\end{aligned}$$

so φ gives an equivalence between $\langle\cdot|\cdot\rangle_1$ and $\langle\cdot|\cdot\rangle_2$, and now it follows that both $3C$-algebras are of the same type.

Conversely, assume that both $3C$-algebras are of the same type and the bilinear forms are equivalent. We take an element $e \in V_1$ with $\langle e|e\rangle_1 \neq 0$, we can choose an element $f \in V_2$ with $\langle f|f\rangle_2 = \langle e|e\rangle_1$. Now, the corresponding composition algebras constructed on V_1 and V_2 by means of Theorem 2 have equivalent norm forms, so they are isomorphic (see [**Sch**]). Let ψ an isomorphism of these composition algebras. If both $3C$-algebras are of type I, then $\psi(\{a, b, c\}_1) = \langle e|e\rangle_1 \psi((ab)c) = \langle f|f\rangle_2 ((\psi(a)\overline{\psi(b)})\psi(c)) = \{\psi(a), \psi(b), \psi(c)\}$, so ψ is actually an isomorphism of $3C$-algebras. The same happens in case they are of type II. □

If we change (2), so as to impose $\langle X(a_1, a_2, a_3)|X(a_1, a_2, a_3)\rangle = \alpha \det(\langle a_i|a_j\rangle)$, $0 \neq \alpha \in F$, then we can extend scalars to a certain extension field K such that $\sqrt{\alpha} \in K$ and consider in $V_K = K \otimes_F V$

the new $\tilde{X} = (\sqrt{\alpha})^{-1}X$, so $\tilde{\Phi} = (\sqrt{\alpha})^{-1}\Phi$. Then,

$$\langle \tilde{X}(a_1,a_2,a_3)|\tilde{X}(b_1,b_2,b_3)\rangle =$$
$$\det(\langle a_i|b_j\rangle + \epsilon \sum_{\sigma \text{ even}} \sum_{\tau \text{ even}} \langle a_{\sigma(1)}|b_{\tau(1)}\rangle \tilde{\Phi}(a_{\sigma(2)},a_{\sigma(3)},b_{\tau(2)},b_{\tau(3)}),$$

so

$$\langle X(a_1,a_2,a_3)|X(b_1,b_2,b_3)\rangle =$$
$$\alpha\det(\langle a_i|b_j\rangle + \epsilon\sqrt{\alpha} \sum_{\sigma \text{ even}} \sum_{\tau \text{ even}} \langle a_{\sigma(1)}|b_{\tau(1)}\rangle \Phi(a_{\sigma(2)},a_{\sigma(3)},b_{\tau(2)},b_{\tau(3)}),$$

with $\epsilon = 0, \pm 1$. In case the dimension is 8, we may consider elements in V so that the last sum is nonzero and we conclude that $\sqrt{\alpha} \in F$. Moreover, with $\beta^2 = \alpha$, multiplying the equation $\langle X(a_1,a_2,a_3)|X(a_1,a_2,a_3)\rangle = \alpha\det(\langle a_i|a_j\rangle)$ by β and considering $\langle\cdot|\cdot\rangle' = \beta\langle\cdot|\cdot\rangle$, we obtain (2) with $\langle\cdot|\cdot\rangle$ changed to $\langle\cdot|\cdot\rangle'$. Hence, we may assume always $\alpha = 1$ as in (2).

In defining quaternionic and octonionic triple systems, Okubo considers a finite–dimensional vector space equipped with a nondegenerate symetric bilinear form $\langle\cdot|\cdot\rangle$ and a nonzero triple product

$$[\cdot,\cdot,\cdot] : V \times V \times V \longrightarrow V$$
$$(a,b,c) \mapsto [a,b,c]$$

verifying that $\langle a|[b,c,d]\rangle$ is skew–symmetric (that is, $[\cdot,\cdot,\cdot]$ satisfies (1)) and the condition

$(\tilde{9})$
$$\langle [a_1,a_2,a_3]|[b_1,b_2,b_3]\rangle =$$
$$\alpha\det(\langle a_i|b_j\rangle) + \beta \sum_{\sigma \text{ even}} \sum_{\tau \text{ even}} \langle a_{\sigma(1)}|b_{\tau(1)}\rangle \langle a_{\sigma(2)}|[a_{\sigma(3)},b_{\tau(2)},b_{\tau(3)}]\rangle$$

The paragraph above shows that if $\alpha \neq 0$ and $\dim V = 8$, necessarily $\alpha \in F^2$ and $\beta^2 = \alpha$. Moreover, $(\tilde{9})$ is then a consequence of

$$\langle [a_1,a_2,a_3]|[a_1,a_2a_3,\rangle] = \alpha\det(\langle a_i|a_j\rangle).$$

On the other hand, if $\dim V = 4$, β is multiplied by 0, so the last summand in $(\tilde{9})$ is superfluous. In case $\dim V \leq 2$, everything is trivial.

On the contrary, if $\alpha = 0$ and $\beta \neq 0$, take $e \in V$ with $\langle e|e\rangle \neq 0$, $W = (Fe)^{\perp}$ and define an anticommutative product in W by $x \cdot y = [x, y, e]$. The nondegeneracy of $\langle\cdot|\cdot\rangle$ and $(\tilde{9})$ gives for $x, y, z \in W$:

$$(x \cdot y) \cdot z = [[x, y, e], z, e] = \beta\langle x|[y, z, e]\rangle e + \beta\langle e|e\rangle[x, y, z],$$

so

$$(x \cdot y) \cdot z = \beta\langle x|y \cdot z\rangle e + \beta\langle e|e\rangle[x, y, z],$$

which is skew-symmetric in x, y, z.

If x_1, x_2, x_3 are orthogonal non isotropic vectors in W, $((x_1 \cdot x_2) \cdot x_3) \cdot x_4$ is skew-symmetric on its arguments and by $(\tilde{9})$:

$$\begin{aligned}((x_1 \cdot x_2) \cdot x_3) \cdot x_4 &= \beta\langle e|e\rangle[[x_1, x_2, x_3], x_4, e] \\ &= \beta\langle e|e\rangle \sum_{\sigma \text{ even}} \langle x_{\sigma(1)}|x_{\sigma(2)} \cdot x_4\rangle x_{\sigma(3)}.\end{aligned}$$

By skew symmetry, x_4 should play the same role than x_1, x_2, x_3, so $0 = ((x_1 \cdot x_2) \cdot x_3) \cdot x_4 = \langle x_{\sigma(1)}|x_{\sigma(2)} \cdot x_4\rangle$. As a consequence, $\langle V|x \cdot y\rangle = 0$ for any orthogonal $x, y \in W$ and this implies $[x, y, e] = 0$ for any $x, y, e \in V$, so $[\cdot,\cdot,\cdot] = 0$. Therefore, $\dim W \leq 3$, $\dim V \leq 4$ and then $(\tilde{9})$ reduces to $[[x, y, z], t, u] = 0$ for any $x, y, z, t, u \in V$. Of course, in case $\alpha = \beta = 0$ again $(\tilde{9})$ reduces to $[[x, y, z], t, u] = 0$. This proves equations (2.6) in [**O**].

3. Automorphisms and derivations

The objective in this section is to determine the automorphism group and derivation algebra of any $3C$-product. For dimension 8, this has been done over $\mathbb{R}$ in [**Sh3**], by working inside the Lie algebra $so(V, \langle\cdot|\cdot\rangle)$ of skew-symmetric transformations relative to $\langle\cdot|\cdot\rangle$ and considering a distinguished seven-dimensional subspace.

Our approach is to consider exactly the same spin representation as in [**E–M**], so that everything works very smoothly by using Moufang identities. For lower dimension, things are easier. First we reformulate some previous remarks in:

LEMMA 5. *Let $(V, \langle\cdot|\cdot\rangle, \{\cdot,\cdot,\cdot\})$ be a 3C–algebra, X the associated three–fold vector cross product as in Proposition 1, Φ the corresponding skew–symmetric 4–linear form as in Proposition 3, and let $\varphi \in GL(V)$. Then, φ is an automorphism of $(V, \{\cdot,\cdot,\cdot\})$ if and only if φ is orthogonal relative to $\langle\cdot|\cdot\rangle$ ($\varphi \in O(V, \langle\cdot|\cdot\rangle)$) and preserves Φ ($\Phi(\varphi(a), \varphi(b), \varphi(c), \varphi(d)) = \Phi(a,b,c,d)$ for any $a,b,c,d \in V$).*

COROLLARY 6 [**Sh2**]. *Let $(V, \langle\cdot|\cdot\rangle, \{\cdot,\cdot,\cdot\})$ be a 3C–algebra. Then,*

i) *If $\dim V = 1$ or 2, then $\mathrm{Aut}(V, \{\cdot,\cdot,\cdot\}) = O(V, \langle\cdot|\cdot\rangle)$.*

ii) *If $\dim V = 4$, then $\mathrm{Aut}(V, \{\cdot,\cdot,\cdot\}) = SO(V, \langle\cdot|\cdot\rangle)$ (the special orthogonal group).*

Proof. i) is clear and notice that if $\dim V = 4$ and $\varphi \in GL(V)$, then

$$\Phi(\varphi(a), \varphi(b), \varphi(c)\varphi(d)) = (\det\varphi)\Phi(a,b,c,d)$$

for any a,b,c,d by the skew–symmetry of Φ. $\square$

Now, assume that the dimension of V is 8 and fix an element $e \in V$ with $\langle e|e\rangle \neq 0$. By Theorem 2, we may assume that there is a binary multiplication xy on V, with identity element e, such that $\langle xy|xy\rangle_1 = \langle x|x\rangle_1\langle y|y\rangle_1$, for $\langle\cdot|\cdot\rangle_1 = \langle e|e\rangle^{-1}\langle\cdot|\cdot\rangle$, and $\{a,b,c\} = \langle e|e\rangle(a\bar{b})c$ (the same arguments apply for type II). Let us consider the orthogonal subspace $W = (Fe)^{\perp}$ and the quadratic form $q(x) = -\langle x|x\rangle_1$, for any $x \in W$. Also, for any $x \in W$, let L_x denote the left multiplication by x in V: $L_x : V \to V$, $v \mapsto xv$. L_x satisfies $(L_x)^2 = q(x)I$ and, therefore, we get the representation of the Clifford algebra $C(W,q)$ in V as in [**E–M**]:

$$\begin{aligned} \rho_L : C(W,q) &\longrightarrow End_F(V) \\ x &\mapsto \quad L_x. \end{aligned}$$

The restriction of ρ_L to the even Clifford algebra $C^+(W,q)$ is an isomorphism ([**E–M**]). We denote by a dot the multiplication in $C(W,q)$.

Now, the corresponding spin group is (see [**J2**; Theorem 4.14]):

$$Spin(W,q) = \{x_1 \cdot \ldots \cdot x_{2r} : x_i \in W \text{ and } \prod_{i=1}^{2r} q(x_i) = 1\} (\subseteq C^+(W,q))$$

and by the Cartan–Dieudonné Theorem, r can be taken to be 1, 2 or 3.

But for any $x \in W$, using Moufang identities we obtain:

$$\begin{aligned}\{xa, xb, xc\} &= \langle e|e\rangle\big((xa)(\bar{b}\bar{x})\big)(xc) = -\langle e|e\rangle\big((xa)(\bar{b}x)\big)(xc)\\ &= -\langle e|e\rangle(x(a\bar{b})x)(xc) \text{ by middle Moufang}\\ &= -\langle e|e\rangle x\big((a\bar{b})(x(xc))\big) \text{ by left Moufang}\\ &= -q(x)\langle e|e\rangle x\big((a\bar{b})c\big) = -q(x)x\{a, b, c\}.\end{aligned}$$

This immediately implies

$$\rho_L(Spin(W, q) \subseteq \mathrm{Aut}(V, \{\cdot, \cdot, \cdot\}).$$

PROPOSITION 7. *Let $(V, \langle\cdot|\cdot\rangle, \{\cdot, \cdot, \cdot\})$ be a 3C-algebra of dimension 8 and type I, e an element of V with $\langle e|e\rangle \neq 0$, so that $\{a, b, c\} = \langle e|e\rangle(a\bar{b})c$ for any $a, b, c \in V$, for a convenient (binary) composition product on V. Let W and q be as above. Then the automorphism group of the composition algebra V is the isotropy subgroup of* $\mathrm{Aut}(V, \{\cdot, \cdot, \cdot\})$ *at e and it is contained in $\rho_L(Spin(W, q))$.*

Proof. The first assertion is clear. Now, if φ is an automorphism of the composition algebra V, since $\rho_L(C^+(W, q)) = End_F(V)$, we pick up un element $a \in C^+(W, q)$ such that $\rho_L(a) = \varphi$. For any $x \in W$ and $v \in V$:

$$\begin{aligned}\rho_L(a \cdot x \cdot a^{-1})(v) &= \varphi\rho_L(x)\varphi^{-1}(v)\\ &= \varphi(x\varphi^{-1}(v))\\ &= \varphi(x)\varphi(\varphi^{-1}(v))\\ &= \rho_L(\varphi(x))(v)\end{aligned}$$

so, if $q(x) \neq 0$, $1 - \varphi(x)^{-1} \cdot a \cdot x \cdot a^{-1} \in \ker \rho_L|_{C^+(W,q)} = 0$ and $a \cdot x \cdot a^{-1} = \varphi(x)$. As a consequence, $a \cdot x \cdot a^{-1} = \varphi(x)$ for any $x \in W$ and a belongs to the even Clifford group Γ^+, so $a = x_1 \cdot \ldots \cdot x_{2r}$, with $x_i \in W$ and $q(x_i) \neq 0$ for any i. But $\rho_L(a)(e) = \varphi(e) = e$, so $x_1(\cdots(x_{2r}e)\cdot\cdot) = e$, $x_1(x_2\cdots(x_{2r-1}x_{2r})\cdot\cdot) = e$. Hence, $q(x_1)\cdots q(x_{2r}) = 1$ and $a \in Spin(W, q)$. □

As a byproduct, we obtain the following description of the automorphism group of any Cayley–Dickson algebra:

COROLLARY 8. *Let C be any Cayley–Dickson algebra with norm n and C_0 the set of trace zero elements. Then,* $\mathrm{Aut}\, C = \{L_{x_1} L_{x_2} \cdots L_{x_{2r}} : x_i \in C_0,\ \prod_{i=1}^{2r} n(x_i) = 1 \quad \textit{and} \quad x_1(x_2 \cdots (x_{2r-1}x_{2r}) \cdots) = 1\}$. □

Using the conjugation, or working with the $3C$–algebra of type II given by $\{a, b, c\} = a(\bar{b}c)$ defined on the Cayley–Dickson algebra C, we can substitute left by right multiplications in the Corollary above.

REMARK. Corollary 8, together with the argument in [**A–H**; Section 10, Example (2)], shows that, given a Cayley–Dickson algebra C, its structure group as a structurable algebra is precisely $\rho_L(\Gamma^+)$ (same notation as in Proposition 7).

LEMMA 9. *$Spin(W, q)$ acts transitively (by means of ρ_L) on* $\{v \in V : \langle v|v\rangle_1 = 1\} = \{v \in V : \langle v|v\rangle = \langle e|e\rangle\}$.

Proof. It is enough to see that for any $v \in V$ with $\langle v|v\rangle_1 = 1$, there is an element $a \in Spin(W, q)$ with $\rho_L(a)(e) = v$. For this, consider the left multiplication $L_v : V \to V$, which is nonsingular, and a nonisotropic element $x \in L_v^{-1}(W) \cap W$ (subspace of dimension at least 6). Then, $L_v(x) = vx \in W$, so $(vx)x = q(x)v$ and with $x_1 = vx$, $x_2 = q(x)^{-1}x$, we obtain $v = x_1x_2$, so $1 = \langle v|v\rangle_1 = \langle x_1|x_1\rangle_1\langle x_2|x_2\rangle_1 = q(x_1)q(x_2)$, $x_1 \cdot x_2 \in Spin(W, q)$ and $v = \rho_L(x_1 \cdot x_2)(e)$. □

THEOREM 10. *Let $(V, \langle\cdot|\cdot\rangle, \{\cdot,\cdot,\cdot\})$ be a $3C$–algebra of dimension 8 and type I, let e be any element of V with $\langle e|e\rangle \neq 0$, $W = (Fe)^\perp$ and $q(x) = -\langle e|e\rangle^{-1}\langle x|x\rangle$ for any $x \in W$. Then,* $\mathrm{Aut}(V, \{\cdot,\cdot,\cdot\}) = \rho_L(Spin(W, q))(\cong Spin(W, q))$.

Proof. For any $\varphi \in \mathrm{Aut}(V, \{\cdot,\cdot,\cdot\})$, $\langle\varphi(e)|\varphi(e)\rangle = \langle e|e\rangle$, so by Lemma 9, there is an element $a \in Spin(W, q)$ such that $\varphi(e) = \rho_L(a)(e)$. Thus, $\psi = \rho_L(a)^{-1}\varphi \in \mathrm{Aut}(V, \{\cdot,\cdot,\cdot\})$ and fixes e. By proposition 7, $\psi \in \rho_L(Spin(W, q))$ and so does φ. □

REMARK. For type II $3C$–algebras, everything works in the same way but dealing with the representation given by right multiplications $\rho_R : C(W, q) \to End_F(V)$, $x \mapsto R_x$. As representations of $C^+(W, q)$, ρ_L and ρ_R are equivalent, but they are not as representations of $C(W, q)$ (see [**E–M**]).

As for derivations, if $(V, \langle\cdot|\cdot\rangle, \{\cdot,\cdot,\cdot\})$ is a $3C$–algebra and $\varphi \in End_F(V)$, $\varphi \in \mathrm{Der}(V, \{\cdot,\cdot,\cdot\})$ if and only if φ is skew–symmetric relative to

$\langle\cdot|\cdot\rangle$ (that is, $\varphi \in so(V, \langle\cdot|\cdot\rangle)$) and

$$\Phi(\varphi(a), b, c, d) + \Phi(a, \varphi(b), c, d) + \Phi(a, b, \varphi(c), d) + \Phi(a, b, c, \varphi(d)) = 0$$

for any $a, b, c, d \in V$. Therefore,

PROPOSITION 11. *If $(V, \langle\cdot|\cdot\rangle, \{\cdot,\cdot,\cdot\})$ is a 3C algebra and the dimension of V is ≤ 4, then $\mathrm{Der}(V, \{\cdot,\cdot,\cdot\}) = so(V, \langle\cdot|\cdot\rangle)$.* □

So let us consider a 3C algebra $(V, \langle\cdot|\cdot\rangle, \{\cdot,\cdot,\cdot\})$ with $\dim V = 8$ and type I, we fix an element $e \in V$ with $\langle e|e\rangle \neq 0$, $W = (Fe)^{\perp}$ and (W, q) as above. For any $x \in W$ and $a, b, c \in V$:

$$\begin{aligned} x\big((ab)c\big) &= -(x, ab, c) + \big(x(ab)\big)c \\ &= (ab, x, c) - (x, a, b)c + \big((xa)\bar{b}\big)c \\ &= \big((ab)x\big)c - (ab)(xc) - (a, \bar{b}, x)c + \big((xa)\bar{b}\big)c \\ &= \big((ab)x\big)c - (ab)(xc) - \big((a\bar{b})x\big)c + \big(a(\bar{b}x)\big)c + \big((xa)b\big)c \\ &= \big((xa)b\big)c - \big(a(\overline{xb})\big)c - (ab)(xc), \end{aligned}$$

where $(a, b, c) = (ab)c - a(bc)$ is the associator of the elements a, b, c in the (binary) composition algebra V. Therefore:

$$L_x\{a, b, c\} = \{L_x a, b, c\} - \{a, L_x b, c\} - \{a, b, L_x c\}. \tag{10}$$

Thus, for any $x, y \in W$:

$$\begin{aligned} [L_x, L_y]\{a, b, c\} &= L_x\big(\{L_y a, b, c\} - \{a, L_y b, c\} - \{a, b, L_y c\}\big) \\ &\qquad - L_y\big(\{L_x a, b, c\} - \{a, L_x b, c\} - \{a, b, L_x c\}\big) \\ &= \cdots \text{ apply (10) six times} \\ &= \{[L_x, L_y]a, b, c\} + \{a, [L_x, L_y]b, c\} + \{a, b, [L_x, L_y]c\}, \end{aligned}$$

and $[L_x, L_y] \in \mathrm{Der}(V, \{\cdot,\cdot,\cdot\})$. Moreover, for any $x \in W$ and $a, b \in V$

$$\langle xa|b\rangle_1 = \langle a|\bar{x}b\rangle_1 = -\langle a|xb\rangle_1,$$

so $L_x \in so(V, \langle\cdot|\cdot\rangle)$.

We consider in $C(W,q)$ the subspace $[W,W]^{\cdot} = \text{span}\{[x,y]^{\cdot} f = x\cdot y - y\cdot x : x,y \in W\}$ and the representation $\rho_L : C(W,q) \to End_F(V)$ as above. Since the restriction of ρ_L to $C^+(W,q)$ is an isomorphism, the restriction $\rho_L|_{[W,W]^{\cdot}}$ is one–one. In case the restriction of ρ_L to $W \oplus [W,W]^{\cdot} (\subseteq C(W,q))$ were not one–one, there would exist some element $x \in W$ such that $\rho_L(x) \in \rho_L([W,W]^{\cdot}) \subseteq \text{Der}(V,\{\cdot,\cdot,\cdot\})$, so by (10) $\{a,xb,c\} + \{a,b,xc\} = 0$ for any $a,b,c \in V$. With $b = e$ this shows $(ax)c = a(xc)$ for any $a,c \in V$ and x belongs to the nucleus of the Cayley–Dickson algebra V, which is Fe. Thus, $x \in Fe \cap W = 0$. Hence, by dimension counting

$$so(V,\langle\cdot|\cdot\rangle) = \rho_L(W) \oplus \rho_L([W,W]^{\cdot})$$

and

$$\begin{aligned} &\rho_L([W,W]^{\cdot}) \subseteq \text{Der}(V,\{\cdot,\cdot,\cdot\}) \subseteq so(V,\langle\cdot|\cdot\rangle), \\ &\rho_L(W) \cap \text{Der}(V,\{\cdot,\cdot,\cdot\}) = 0. \end{aligned}$$

Therefore,

THEOREM 12. *Let $(V,\langle\cdot|\cdot\rangle,\{\cdot,\cdot,\cdot\})$ be a $3C$–algebra of dimension 8 and type I, let e be any element of V with $\langle e|e\rangle \neq 0$, $W = (Fe)^{\perp}$ and $q(x) = -\langle e|e\rangle^{-1}\langle x|x\rangle$ for any $x \in W$. Then,* $\text{Der}(V,\{\cdot,\cdot,\cdot\}) = \rho_L([W,W]^{\cdot}) = \text{span}\{[L_x,L_y] : x,y \in W\} (\cong so(W,q))$. *Moreover, the derivation algebra of the corresponding binary composition algebra is* $\text{Der}\, V = \{d \in \text{Der}(V,\{\cdot,\cdot,\cdot\}); d(e) = 0\}$. $\square$

This gives another description of the derivation algebra of Cayley–Dickson algebras valid over any field of characteristic $\neq 2$ and shows that all the derivations are inner (see [**Sch**; Corollary 3.29]), even in characteristic 3, although in this case the derivation algebra is not simple. It also gives an alternative and coordinate–free proof of Theorem 7, valid also in characteristic 3.

REMARK. Jacobson proved in [**J1**] that given a composition algebra C with symmetric bilinear form $\langle\cdot|\cdot\rangle$, C_0 the subspace of trace zero elements, then:

$$\begin{aligned} \text{Aut}\, C = \{&\varphi \in GL(C) : \varphi(1) = 1, \varphi(C_0) = C_0 \text{ and} \\ &\Lambda(\varphi(x),\varphi(y),\varphi(z)) = \Lambda(x,y,z)\,\forall x,y,z \in C_0\}, \end{aligned}$$

where $\Lambda(x,y,z) = \langle x|y\times z\rangle$ and $y\times z = yz + \langle y|z\rangle 1$ is the projection of xy onto C_0. Λ is a skew-symmetric form in C_0. Similarly,

$$\begin{aligned}\operatorname{Der} C = \{\varphi \in End_F(C) : \varphi(1) = 0, \varphi(C_0) \subseteq C_0 \quad \text{and} \\ \Lambda(\varphi(x),y,z) + \Lambda(x,\varphi(y),z) + \Lambda(x,y,\varphi(z)) = 0 \\ \forall x,y,z \in C_0\}.\end{aligned}$$

If we consider the $3C$ algebra structure on C defined by $\{a,b,c\} = (ab)c$ and the associated three-fold vector cross product and related Φ, then $X(x,1,y) = \{x,1,y\} + \langle x|y\rangle 1 = xy + \langle x|y\rangle 1 = x\times y$ for any $x,y\in C_0$, so $\Lambda(x,y,z) = \Phi(x,y,1,z) = \Phi(1,x,y,z)$ for any $x,y,z\in C_0$, and our results fit smoothly in Jacobson's results.

4. Some trace formulas

Assume in this section that C is a Cayley–Dickson algebra over the field F (remember, characteristic $\neq 2$), with associated nondegenerate symmetric bilinear form $\langle\cdot|\cdot\rangle$ so that

$$x^2 - 2\langle x|1\rangle x + \langle x|x\rangle 1 = 0$$

for any $x\in C$, and define the trace as usual by $\operatorname{tr}(x) = \langle x|1\rangle$. We consider the $3C$ algebra $(C,\langle\cdot|\cdot\rangle,\{\cdot,\cdot,\cdot\})$ of type I, where $\{x,y,z\} = (x\bar{y})z$ for any $x,y,z\in C$, with associated three-fold vector cross product X and skew-symmetric form Φ as in Propositions 1 and 3. Let $C_0 = (F1)^\perp$ its subset of trace zero elements. Then, for any $x,y,z\in C_0$:

$$\begin{aligned}\Phi(1,x,y,z) &= \langle 1|X(x,y,z)\rangle = \operatorname{tr} X(x,y,z) \\ &= \operatorname{tr}\big((x\bar{y})z\big) = -\operatorname{tr}\big(x(yz)\big).\end{aligned} \tag{11}$$

Also, if x,y are orthogonal elements of C_0, $xy+yx=0$, so $X(x,1,y) = xy = \frac{1}{2}[x,y]$. Hence, for any mutually orthogonal $x_1,x_2,x_3,x_4\in C_0$:

$$\begin{aligned}\operatorname{tr}\big(x_1(x_2(x_3x_4))\big) &= \operatorname{tr}\big((x_1x_2)(x_3x_4)\big) \\ &= -\langle x_1x_2|x_3x_4\rangle = -\langle X(1,x_1,x_2)|X(1,x_3,x_4)\rangle \\ &= -\Phi(x_1,x_2,x_3,x_4),\end{aligned} \tag{12}$$

by applying (9) with $\epsilon = 1$. Also, with $x_1 = 1$ and x_2, x_3, x_4 mutually orthogonal in C_0:

$$\begin{aligned}\operatorname{tr}\big(x_1(x_2(x_3x_4))\big) &= -\langle x_2|x_3x_4\rangle \\ &= -\Phi(x_2,x_3,1,x_4) = -\Phi(x_1,x_2,x_3,x_4),\end{aligned}$$

and the same happens with $x_i = 1$ for $i = 2,3$ or 4 and the x_j's mutually orthogonal for $j \neq i$. Therefore, if the characteristic is $\neq 2,3$, since Φ is skew-symmetric:

$$\text{skew}\ \operatorname{tr}\big(x_1(x_2(x_3x_4))\big) = -\Phi(x_1,x_2,x_3,x_4) \tag{13}$$

for any $x_1, x_2, x_3, x_4 \in C$, where

$$\text{skew}\ f(x_1,x_2,x_3,x_4) = \frac{1}{4!}\sum_\sigma \epsilon_\sigma f(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)},x_{\sigma(4)}),$$

with $\epsilon_\sigma = \pm 1$ according to σ being an even or odd permutation.

Now, if we define

$$\begin{aligned}M : C \times C &\longrightarrow C_0 \\ (x,y) &\mapsto \frac{1}{2}[x,y] + \operatorname{tr}(x)y - \operatorname{tr}(y)x,\end{aligned}$$

and $\Psi : C^4 \to F$ by

$$\Psi(x_1,x_2,x_3,x_4) = \operatorname{tr}\big(M(x_1,x_2)M(x_3x_4)\big)$$

as in [**Schw**; (2.7)], $M(1,y) = 1y = -M(y,1)$ for any $y \in C_0$ and $M(x,y) = xy = -M(y,x)$ for any mutually orthogonal $x, y \in C_0$. Therefore:

$$\begin{aligned}\Psi(x_1,x_2,x_3,x_4) &= \operatorname{tr}\big((x_1x_2)(x_3x_4)\big) \\ &= \operatorname{tr}\big(x_1(x_2(x_3x_4))\big) \\ &= -\Phi(x_1,x_2,x_3,x_4)\end{aligned}$$

for mutually orthogonal x_1, x_2, x_3, x_4 in C_0. Again, if some $x_i = 1$ (say $i = 2$) and mutually orthogonal x_j's ($j \neq i$) in C_0:

$$\Psi(x_1,1,x_3,x_4) = -\operatorname{tr}(x_1(x_3x_4)) = \Phi(1,x_1,x_3,x_4) = -\Phi(x_1,x_2,x_3,x_4)$$

too. Thus, again, for characteristic $\neq 2,3$, we obtain

$$\Phi = -\operatorname{skew}\Psi. \tag{14}$$

Now, because of (12), (13), (14), Theorems (3.23) and (3.26) in **[Schw]** may be interpreted as follows:

THEOREM 13. *Let C be the Cayley–Dickson algebra over the complex field. Then,*

i) *The invariants of* $\operatorname{Aut} C$ *are generated by (the polynomial mappings induced by) the restrictions of* $\langle\cdot|\cdot\rangle$, $\Phi(1,\cdot,\cdot,\cdot)$ *and* $\Phi(\cdot,\cdot,\cdot,\cdot)$ *to* C_0.

ii) *The invariants of* $\operatorname{Aut}(C,\langle\cdot|\cdot\rangle,\{\cdot,\cdot,\cdot\})$ *are generated by* $\langle\cdot|\cdot\rangle$ *and* Φ. □

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Departamento de Matematicas y Computacion
Universidad de La Rioja
26004 Logro no, Spain
e-mail: aelduque@msf.unizar.es