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ONE–SIDED BEST SIMULTANEOUS L_1–APPROXIMATION

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1. Introduction

Let X be a compact Hausdorff space, $C(X)$ denote the set of all continuous real valued functions on X and $\ell \in \mathbb{N}$ be any natural number. Define a norm on the space of all ℓ-tuples of elements of $C(X)$ as follows : for any $f_1, \cdots, f_\ell$ in $C(X)$, let $F = (f_1, \cdots, f_\ell)$ and

$$\|F\| = \|(f_1, \cdots, f_\ell)\| = \max_{\mathbf{a} \in A} \|\sum_{i=1}^{\ell} a_i f_i\|_1 \tag{1.1}$$

where $A = \{\mathbf{a} = (a_1, \cdots, a_\ell) \mid \sum_{i=1}^{\ell} a_i = 1,\ a_i \geq 0,\ i = 1, \cdots, \ell\}$.

Now suppose that $F = (f_1, \cdots, f_\ell)$ in $C(X)$ are given and S is an n-dimensional subspace of $C(X)$. We want to consider the problem of approximating these functions simultaneously by elements in

$$S(F) \ := \ \bigcap_{i=1}^{\ell} S(f_i) \ := \ \bigcap_{i=1}^{\ell} \{f \in S : f \leq f_i\}$$

in the sense of the minimization of the norm in (1.1). In other words, we want to find $f \in S(F)$ to minimize

$$\|(f_1 - f, \cdots, f_\ell - f)\|. \tag{1.2}$$

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If such a function f^* exists, it is called a one-sided best simultaneous L_1-approximation of $F = (f_1, \cdots, f_\ell)$. Unless explicitly stated otherwise we shall assume throughout this article that F is an ℓ-tuple of $f_1, \cdots, f_\ell$.

It should be remarked that in this paper we only consider the problem of best approximation from below. The problem of best approximation from above, where the approximating set is

$$M(F) := \bigcap_{i=1}^{\ell} M(f_i) := \bigcap_{i=1}^{\ell} \{g \in M : f_i \leq g\},$$

can be treated in a similar way, where M is an n-dimensional subspace of $C(X)$.

Let μ be a finite positive admissible measure defined on X, that is, $\mu(O) > 0$, for every open set $O \subseteq X$.

First, remark that

$$\max_{\mathbf{a} \in A} \| \sum_{i=1}^{\ell} a_i (f_i - f) \|_1 = \max_{1 \leq i \leq \ell} \| f_i - f \|_1 .$$

This follows from the inequalities

$$\begin{aligned} \max_{1 \leq i \leq \ell} \|f_i\|_1 &\leq \max_{\mathbf{a} \in A} \| \sum_{i=1}^{\ell} a_i f_i \|_1 \\ &\leq \max_{\mathbf{a} \in A} \sum_{i=1}^{\ell} a_i \|f_i\|_1 \\ &\leq \max_{1 \leq i \leq \ell} \|f_i\|_1 . \end{aligned}$$

2. Existence and characterization

In this section, we discuss questions of existence and characterization of a one-sided best simultaneous L_1-approximation.

Firstly, by definition, $S(F)$ is a closed convex subset of S. Thus if $S(F)$ is nonempty, then there exists $f \in S(F)$ which minimize (1.2).

When is $S(F)$ a nonempty set ? If for all $i = 1, \cdots, \ell$, $f_i \geq 0$, then $0 \in S(F)$. If S contains a strictly positive function (or equivalently a strictly negative function), then $S(F)$ is nonempty. We have therefore proven that $S(F)$ is nonempty for every F if and only if S contains a strictly positive function. Even if S does not contain a strictly positive function, we can still consider $f \in S(F)$ to minimize (1.2). However, we shall restrict ourselves to those F which $S(F)$ is nonempty.

Now that we have dealt with the problem of existence, let us turn to the question of characterizing one-sided best simultaneous L_1-approximation.

LEMMA 2.1. *The following statements are equivalent:*
(1) $f^* \in S(F)$ *attains the supremum in*

$$\sup_{f \in S(F)} \int_X f d\mu.$$

(2) f^* *is a one-sided best simultaneous* L_1*-approximation of* F.

Proof. Suppose that $f^* \in S(F)$ and $\int_X f d\mu \leq \int_X f^* d\mu$ for all $f \in S(F)$. Then

$$\begin{aligned}
\|(f_1 - f, \cdots, f_\ell - f)\| &= \max_{\|\mathbf{a}\|_1 = 1} \|\sum_{i=1}^{\ell} a_i(f_i - f)\|_1 \\
&= \max_{\|\mathbf{a}\|_1 = 1} \int_X \sum_{i=1}^{\ell} a_i f_i d\mu - \int_X f d\mu \\
&\geq \max_{\|\mathbf{a}\|_1 = 1} \int_X \sum_{i=1}^{\ell} a_i f_i d\mu - \int_X f^* d\mu \\
&= \|(f_1 - f^*, \cdots, f_\ell - f^*)\|
\end{aligned}$$

for any $f \in S(F)$.

Conversely, if f^* attains the minimization of (1.2), then $\int_X f d\mu \leq \int_X f^* d\mu$ for all $f \in S(F)$, by the above inequality. □

REMARK. Lemma 2.1 yields the following weaker version which is Pinkus' result. Let $\ell = 1$. For each $f \in C(X)$, $u^* \in S(f)$ attaining the infimum in $\inf_{u \in S(f)} \|f-u\|_1$ is equivalent to u^* attaining the supremum in

$$\sup_{u \in S(f)} \int_X u d\mu.$$

THEOREM 2.2. *Suppose that $\int_X f d\mu \neq 0$ for some $f \in S$ and $f_1, \cdots, f_\ell$ in $C(X)$ are given such that there exists $f_0 \in S$ for which $f_0 < f_i$ on X, $i = 1, \cdots, \ell$. Then f^* is a one-sided best simultaneous L_1-approximation of F if and only if $f^* \in S(F)$ and for any $f \in S$ with $f \leq 0$ on $\bigcup_{i=1}^{\ell} Z(f_i - f^*)$, it follows that $\int_X f d\mu \leq 0$, where $Z(f_i - f^*) = \{x \in X : f_i(x) = f^*(x)\}$.*

Proof. Suppose that there exists a $\tilde{f} \in S$ with $\tilde{f} \leq 0$ on $\bigcup_{i=1}^{\ell} Z(f_i - f^*)$ but $\int_X \tilde{f} d\mu > 0$. If $\bigcup_{i=1}^{\ell} Z(f_i - f^*)$ is empty, then there exist $\varepsilon > 0$ and $f \in S$ such that

$$\int_X f d\mu > 0,$$

$$f^* + \varepsilon f \in S(F)$$

and

$$\int_X (f^* + \varepsilon f) d\mu > \int_X f^* d\mu.$$

It is a contradiction. Now we assume that $\bigcup_{i=1}^{\ell} Z(f_i - f^*)$ is nonempty. By assumption, there exists $f_0 \in S$ such that $f_0 < f_i$ on X, $i = 1, \cdots, \ell$. This implies that

$$\tilde{g} := f^* - f_0 > 0 \qquad \text{on} \quad \bigcup_{i=1}^{\ell} Z(f_i - f^*).$$

Then there exists $\delta > 0$ such that

$$\tilde{h} := \tilde{f} - \delta \tilde{g} < 0 \qquad \text{on} \quad \bigcup_{i=1}^{\ell} Z(f_i - f^*)$$

and

$$\int_X \tilde{h} d\mu = \int_X \tilde{f} d\mu - \delta \int_X \tilde{g} d\mu > 0.$$

For each $f \in C(X)$, define $J(f) = \{x \in X : f(x) < 0\}$. Then $\bigcup_{i=1}^{\ell} Z(f_i - f^*) \subset J(\tilde{h})$ and $J(\tilde{h})$ is a proper subset of X since $\int_X \tilde{h} d\mu > 0$. Since $X \setminus J(\tilde{h})$ is compact, there exists $m > 0$ such that $m \le f_i - f^*$ on $X \setminus J(\tilde{h})$ for all $i = 1, \cdots, \ell$. And let $M > 0$ be such that $\tilde{h} \le M$ on X. Let $\varepsilon = m/M$. Then $\varepsilon > 0$, $k := f^* + \varepsilon \tilde{h} \in S(F)$ and

$$\int_X k d\mu > \int_X f^* d\mu,$$

which is a contradiction.

Conversely, we may assume that $\bigcup_{i=1}^{\ell} Z(f_i - f^*)$ is nonempty. Let $f \in S(F)$ and $x \in \bigcup_{i=1}^{\ell} Z(f_i - f^*)$. Then there exists $i \in \{1, \cdots, \ell\}$ such that $f_i(x) = f^*(x)$, so $f(x) \le f_i(x) = f^*(x)$. Thus $f - f^* \le 0$ on $\bigcup_{i=1}^{\ell} Z(f_i - f^*)$. By assumption,

$$\int_X (f - f^*) d\mu \le 0.$$

Hence $\int_X f d\mu \le \int_X f^* d\mu$ for all $f \in S(F)$. By Lemma 2.1, f^* is a one-sided best simultaneous L_1-approximation of F. □

The characterization of Theorem 2.2 is not easy to use. With a little work, we can rewrite Theorem 2.2 in the following more useful form.

LEMMA 2.3 [1]. *Suppose that $\int_X f d\mu \neq 0$ for some $f \in S$. Let K be a closed subset of X with the property that if $f \in S$ satisfies $f(x) \le 0$ on K then $\int_X f d\mu \le 0$. Then there exist $x_1, \cdots, x_k$ in K and positive real numbers $\lambda_1, \cdots, \lambda_k$ such that*

$$\int_X f d\mu = \sum_{i=1}^{k} \lambda_i f(x_i)$$

for all $f \in S$, where $1 \le k \le n$.

THEOREM 2.4. *Suppose that $\int_X f d\mu \neq 0$ for some $f \in S$, and $f_1, \cdots, f_\ell$ in $C(X)$ are given such that there exists f_0 in S for which $f_0 < f_i$ on X, $i = 1, \cdots, \ell$. Then the following are equivalent:*

(1) *f^* is a one-sided best simultaneous L_1-approximation of F.*

(2) *$f^* \in S(F)$ and there exist distinct points $x_1, \cdots, x_k$ in X and positive real numbers $\lambda_1, \cdots, \lambda_k$, $1 \leq k \leq n$ for which*

$$(a) \quad \{x_1, \cdots, x_k\} \subset \bigcup_{i=1}^{\ell} Z(f_i - f^*)$$

$$(b) \quad \int_X f d\mu = \sum_{i=1}^{k} \lambda_i f(x_i) \qquad \text{for all} \quad f \in S.$$

Proof. Suppose that f^* is a one-sided best simultaneous L_1- approximation of F and let $K = \bigcup_{i=1}^{\ell} Z(f_i - f^*)$. Then $f^* \in S(F)$ and there exist $x_1, \cdots, x_k$ in $\bigcup_{i=1}^{\ell} Z(f_i - f^*)$ and positive numbers $\lambda_1, \cdots, \lambda_k$ such that for all $f \in S$

$$\int_X f d\mu = \sum_{i=1}^{k} \lambda_i f(x_i),$$

by Theorem 2.2 and Lemma 2.3.

Conversely, assume that $f^* \in S(F)$ and there exist distinct points $x_1, \cdots, x_k$ in X and positive numbers $\lambda_1, \cdots, \lambda_k$, $1 \leq k \leq n$ for which (a) and (b) hold. Then for any $f \in S(F)$,

$$\begin{aligned} \int_X f d\mu &= \sum_{i=1}^{k} \lambda_i f(x_i) \\ &\leq \sum_{i=1}^{k} \lambda_i f_{j(i)}(x_i) \\ &= \int_X f^* d\mu, \end{aligned}$$

where $j(i) \in \{1, \cdots, \ell\}$ such that $x_i \in Z(f_{j(i)} - f^*)$ for each $i \in \{1, \cdots, k\}$. By Lemma 2.1, f^* is a one-sided best simultaneous L_1-approximation of F. $\square$

COROLLARY 2.5. *Suppose that the conditions of Theorem 2.4 hold. Let f^* be a one-sided best simultaneous L_1-approximation of F with $x_1, \cdots, x_k$ as in Theorem 2.4 (2). Then, for any one-sided best simultaneous L_1-approximation f of F, $f(x_i) = f^*(x_i)$, $i = 1, \cdots, k$.*

Proof. By Lemma 2.1, for any one-sided best simultaneous L_1-approximation f of F, we have

$$\int_X f d\mu = \int_X f^* d\mu,$$

so

$$\int_X f d\mu = \sum_{i=1}^{k} \lambda_i f(x_i) = \sum_{i=1}^{k} \lambda_i f^*(x_i) = \int_X f^* d\mu.$$

Since $\lambda_i > 0$ and $f^*(x_i) - f(x_i) \geq 0$, we obtain $f(x_i) = f^*(x_i)$ for all $i = 1, \cdots, k$. □

Recall that S is linearly independent over $\{x_1, \cdots, x_n\}$ if $f \in S$ and $f(x_i) = 0$, $i = 1, \cdots, n$, then $f = 0$ [1]. Thus we have the following result.

COROLLARY 2.6. *Suppose that the conditions of Theorem 2.4 hold, and let f^* be a one-sided best simultaneous L_1-approximation of F with $x_1, \cdots, x_k$ as in Theorem 2.4 (2). Assume that S is linearly independent over $\{x_1, \cdots, x_k\}$. Then the one-sided best simultaneous L_1-approximation of F is unique.*

The formula (b) of Theorem 2.4 is called a quadrature formula for S. It has the additional property that all the coefficients $\lambda_1, \cdots, \lambda_k$ are positive real numbers. If

$$\int_X f d\mu = \sum_{i=1}^{k} \lambda_i f(x_i) \tag{2.1}$$

for all $f \in S$ where $\lambda_i > 0$, $i = 1, \cdots, k$ and $1 \leq k < \infty$, then we shall say that (2.1) is a positive quadrature formula with k active points $\{x_i\}_{i=1}^k$.

3. Uniqueness

We now turn to the question when we have a unique one-sided best simultaneous L_1-approximation for every ℓ−tuples of elements of $C(X)$.

LEMMA 3.1 [1]. *Let S be an n-dimensional subspace of $C(X)$ and assume that there exists $f \in S$ with $\int_X f d\mu \neq 0$. Let $\{s_1, \cdots, s_n\}$ be any basis for S. If for all $f \in S$*

$$\int_X f d\mu = \sum_{i=1}^{k} \lambda_i f(x_i)$$

where $1 \leq k < \infty$, $\lambda_i > 0$, $i = 1, \cdots, k$ and

$$rank[s_i(x_j)]_{n \times k} < k,$$

then there exists a positive quadrature formula for S with r active points $\{y_1, \cdots, y_r\}$ where $1 \leq r < k$, and $\{y_1, \cdots y_r\} \subset \{x_1, \cdots, x_k\}$.

DEFINITION 3.2. A subspace S of $C(X)$ is said to be a one-sided ℓ-simultaneous L_1-unicity space if for each $F = (f_1, \cdots, f_\ell)$ in $C(X)$, there exists a unique one-sided best simultaneous L_1-approximation of F.

THEOREM 3.3. *Suppose that S contains a strictly positive function and the dimension of S is $n \geq 2$. Then S is a one-sided ℓ-simultaneous L_1-unicity space if and only if each positive quadrature formula for S contains at least n active points.*

Proof. Suppose not, that is, for all $f \in S$

$$\int_X f d\mu = \sum_{i=1}^{k} \lambda_i f(x_i)$$

where $1 \le k \le n-1$ and $\lambda_i > 0, i = 1, \cdots, k$. Since $dimS = n > k$, there exists a $f^* \in S \setminus \{0\}$ satisfying $f^*(x_i) = 0,\ i = 1, \cdots, k$. Set $g_i = |f^*|,\ i = 1, \cdots, \ell$. Let $G = (g_1, \cdots, g_\ell)$. Then $g_i \in C(X),\ \pm f^* \le g_i,\ i = 1, \cdots, \ell$, and

$$(a)\ (g_i \pm f^*)(x_j) = 0 \qquad i = 1, \cdots, \ell,\ j = 1, \cdots, k,$$

$$(b)\ \int_X f d\mu = \sum_{i=1}^{k} \lambda_i f(x_i) \qquad \text{for all } f \in S.$$

By Theorem 2.4, $\pm f^*$ are one-sided best simultaneous L_1-approximat ions of G. This is a contradiction.

Conversely, suppose that there exist $f_1, \cdots, f_\ell$ in $C(X)$ such that there exist one-sided best simultaneous L_1-approximations g_1, g_2 of F. Since $dimS = n$, it follows from Lemma 3.1 that we may assume that for every positive quadrature formula for S with n active points $\{x_1, \cdots, x_n\}$ we have $\det[s_i(x_j)]_{n\times n} \neq 0$, where $\{s_1, \cdots, s_n\}$ is any basis for S. Thus S is linearly independent over $\{x_1, \cdots, x_n\}$. From Corollary 2.5 and our assumption, there exists a positive quadrature formula for S with n active points $\{x_1, \cdots, x_n\}$ and $g_1(x_i) = g_2(x_i),\ i = 1, \cdots, n$, and so $g_1 = g_2$. □

Recall that S is a one-sided L_1-unicity space of $C(X)$ if for every $f \in C(X)$ there exists a unique one-sided best L_1-approximation of f.

DEFINITION 3.4. If for all $\ell \in \mathbb{N}$, S is a one-sided ℓ-simultaneous L_1-unicity space, then S is called a one-sided simultaneous L_1-unicity space.

In this paper, ℓ is any natural number. Then we can rewrite Theorem 3.3 as follows.

COROLLARY 3.5. *Suppose that S contains a strictly positive function and the dimension of S is $n \ge 2$. Then the following are equivalent:*

(1) *S is a one-sided L_1-unicity space of $C(X)$.*
(2) *Each positive quadrature formula for S contains at least n active points.*
(3) *S is a one-sided ℓ-simultaneous L_1-unicity space for some $\ell \in \mathbb{N}$.*
(4) *S is a one-sided simultaneous L_1-unicity space.*

COROLLARY 3.6 [1]. *Let S be an n-dimensional subspace of $C(X)$ with $n \geq 2$. Assume S contains a strictly positive function. Then S is a one-sided L_1-unicity space for $C(X)$ if and only if each positive quadrature formula for S contains at least n active points.*

THEOREM 3.7. *Let S be an n-dimensional subspace of $C(X)$ with $n \geq 2$. Then S is a one-sided simultaneous L_1-unicity space if and only if for every $f \in S \setminus \{0\}$, the zero function is not a one-sided best simultaneous L_1-approximation of the ℓ-tuple $(|f|, \cdots, |f|)$ for any $\ell \in \mathbb{N}$.*

Proof. By Corollary 3.5, it suffices to show that the necessary condition is true for some $\ell \in \mathbb{N}$. Suppose not, that is, there exists $f \in S \setminus \{0\}$ for which the zero function is a one-sided best simultaneous L_1-approximation of $(|f|, \cdots, |f|)$. Since $\pm f \in S((|f|, \cdots, |f|))$, $\int_X \pm f d\mu = 0$. Thus $\pm f$ are one-sided best simultaneous L_1-approximations of $(|f|, \cdots, |f|)$, which is a contradiction.

Conversely, suppose that there exists $F = (f_1, \cdots, f_\ell)$ such that F has distinct one-sided best simultaneous L_1-approximations g_1 and g_2. Set $f^* = (g_1 - g_2)/2$ and $h_i = f_i - (g_1 + g_2)/2$, $i = 1, \cdots, \ell$. Let $H = (h_1, \cdots, h_\ell)$. Then $\pm f^*$ are one-sided best simultaneous L_1-approximations of H. So $|f^*| \in S(H)$. Let $|F^*| = (|f^*|, \cdots, |f^*|)$. Then $S(|F^*|) \subset S(H)$ and $|f^*| \in S(|F^*|)$. Thus $\pm f^*$ are one-sided best simultaneous L_1-approximations of $|F^*|$ and the zero function is a one-sided best simultaneous L_1-approximation of $|F^*|$. It is a contradiction. □

COROLLARY 3.8 [1]. *A subspace S of $C(X)$ is a one-sided L_1-unicity space if and only if for every $f \in S \setminus \{0\}$, the zero function is not a one-sided best L_1-approximation of $|f|$.*

As an immediate consequence of this theorem, we have the following three results.

COROLLARY 3.9. *A finite dimensional subspace S of $C(X)$ is a one-sided simultaneous L_1-unicity space if and only if for each $f \in S \setminus \{0\}$ there exists $g \in S$ satisfying*

(a) $g \leq |f|$,

(b) $\int_X g d\mu > 0$.

COROLLARY 3.10. *A finite dimensional subspace S of $C(X)$ is a one-sided simultaneous L_1-unicity space if and only if S is a one-sided L_1-unicity space.*

COROLLARY 3.11. *Let S be a finite dimensional subspace of $C(X)$ and assume that S contains a strictly positive function. Then S is a one-sided simultaneous L_1-unicity space if and only if for each $f \in S \setminus \{0\}$, there exists $g \in S$ satisfying*

(a) $g \leq 0$ *on* $Z(f)$,

(b) $\int_X g d\mu > 0$.

Note that $int(X)$ denotes the interior of X.

COROLLARY 3.12. *Let S be an n-dimensional subspace of $C(X)$ with $n \geq 2$. Assume that $int(X)$ is connected and there exists $f^* \in S$ such that $f^* > 0$ on $int(X)$. Then S is not a one-sided simultaneous L_1-unicity space.*

Proof. By [1] , S is not a one-sided L_1-unicity space. By Corollary 3.10, S is not a one-sided simultaneous L_1-unicity space. □

The condition that S contains a strictly positive function on $int(X)$ is essential. If $S = \text{span}\{x, |x|\}$ on $X = [-1, 1]$, then for every $f \in S, |f| \in S$, and therefore by Theorem 3.7, S is a one-sided simultaneous L_1-unicity space. The condition that the $int(X)$ is connected is also necessary. To see this, let S be as previously defined, and

$$X = \{(x, y) : |y| \leq |x| \leq 1\}.$$

Again, Theorem 3.7 implies that S is a one-sided simultaneous L_1-unicity space.

Note that $[X]$ denotes the number of connected components of X.

THEOREM 3.13. *Let S be an n-dimensional subspace of $C(X)$, $n \geq 2$. Assume that S contains a strictly positive function and $[X] \leq (n-1)$. Then S is not a one-sided simultaneous L_1-unicity space.*

Proof. By [1] , S is not a one-sided L_1-unicity space. By Corollary 3.5, S is not a one-sided simultaneous L_1-unicity space. □

By the following examples, if $[X] \geq n$, then the above theorem simply does not hold.

EXAMPLE 3.14. (1) Let $X = [0,1] \cup [2,3]$ and $S = \text{span}\{s_1, s_2, s_3\}$ where $s_1 = \chi_{[1,2]}$, $s_2 = x \cdot \chi_{[2,3]}$ and $s_3 = x^2$. Clearly, S contains a strictly positive function on X. Then by Theorem 3.13, S is not a one-sided simultaneous L_1-unicity space. Moreover, S has a positive quadrature formula with two active points $\{1, 34/15\}$ and $\lambda_1 = 1$, $\lambda_2 = 75/68$.

(2) Let $X = \bigcup_{i=1}^{n} A_i$, where $A_1, \cdots, A_n$ are distinct components of X and let s_i be strictly positive on A_i which vanish elsewhere. Set $S = \text{span}\{s_1, \cdots, s_n\}$. Then $[X] = n$. By Theorem 3.7, S is a one-sided simultaneous L_1-unicity space.

(3) Let $S = \text{span}\{1, x\}$ on $X = [-2,-1] \cup [1,2]$ where μ is the Lebesgue measure. Since

$$\int_X 1 d\mu = 2 = 2 \cdot 1(t) \qquad \text{for any} \quad t \in X$$

$$\int_X x d\mu = 0 \neq 2 \cdot x(t) \qquad \text{for any} \quad t \in X,$$

S is a one-sided simultaneous L_1-unicity space since there does not exist a positive quadrature formula for S with one active point in X.

LEMMA 3.15 [1]. *Let S be a finite dimensional subspace of $C(X)$. There exist points $x_1, \cdots, x_m$ such that if $f \in S$ satisfies $f(x_i) \leq 0$, $i = 1, \cdots, m$ and $\int_X f d\mu \geq 0$, then $f = 0$.*

PROPOSITION 3.16. *If S is a finite dimensional subspace of $C(X)$, then the set*

$$\{F : F \text{ has a unique one-sided best simultaneous } L_1\text{-approximation}\}$$

is dense in $\{F : S(F) \text{ is nonempty}\}$ with L_1-norm.

Proof. Let $F = (f_1, \cdots, f_\ell)$ be with $S(F) \neq \phi$. There exist points $x_1, \cdots, x_m$ such that if $f \in S$ satisfies $f(x_i) \leq 0$, $i = 1, \cdots, m$ and $\int_X f d\mu \geq 0$ then $f = 0$. Choose any $h \in S(F)$. Given $\varepsilon > 0$, for all $i = 1, \cdots, \ell$, let $g_i \in C(X)$ satisfy $h \leq g_i$, $h(x_j) = g_i(x_j)$, $j = 1, \cdots, m$, and

$$\|f_i - g_i\|_1 < \varepsilon/\ell.$$

Let $G = (g_1, \cdots, g_\ell)$. Then

$$\begin{aligned} \|F - G\| &= \max_{\|\mathbf{a}\|_1 = 1} \|\sum_{i=1}^{\ell} a_i(f_i - g_i)\|_1 \\ &\le \sum_{i=1}^{\ell} \|f_i - g_i\|_1 \\ &< \varepsilon. \end{aligned}$$

We claim that G has a unique one-sided best simultaneous L_1-approximation. By our construction, $h \in S(G)$. If k is a one-sided best simultaneous L_1-approximation of G, then for all $i = 1, \cdots, \ell$,

$$k(x_j) \le g_i(x_j) = h(x_j) \qquad j = 1, \cdots, m,$$
$$\int_X k d\mu \ge \int_X h d\mu.$$

Set $w = k - h$. Then $w(x_i) \le 0,\ i = 1, \cdots, m,$ and $\int_X w d\mu \ge 0$. From Lemma 3.15, $w = 0$. Thus $k = h$. Hence G has a unique one-sided best simultaneous L_1-approximation h. □

COROLLARY 3.17 [1]. *If S is a finite dimensional subspace of $C(X)$, then the set $\{f : f$ has a unique one-sided best L_1- approximation$\}$ is dense in $\{f : S(f) \ne \phi\}$ with L_1-norm.*

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