

PRODUCT SPACES THAT INDUCE APPROXIMATE FIBRATIONS

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0. Introduction

In the study of manifold decompositions, a central theme is to understand the source manifold taking advantage of the informations of a base space and a decomposition. The concepts of both Hurewicz fibrations and cell-like maps have played very important roles for investigating the mutual relations of three objects. But it is somewhat restrictive for a decomposition map to be cell-like because its inverse images must have trivial shapes.

Coram and Duvall [1] introduced an approximate fibration as a map having the approximate homotopy lifting property for every space, which is a generalization of a Hurewicz fibration and a cell-like map but has as valuable properties as well Hurewicz fibration and is widely applicable to the maps whose fibers are nontrivial shapes.

A proper map $p : M \rightarrow B$ between locally compact ANR's is called an *approximate fibration* if it has the following approximate homotopy lifting property: given an open cover ε of B , an arbitrary space X , and two maps $g : X \rightarrow M$ and $F : X \times I \rightarrow B$ such that $p \circ g = F_0$, there exists a map $G : X \times I \rightarrow M$ such that $G_0 = g$ and $p \circ G$ is ε -close to F .

We assume all spaces are locally compact, metrizable ANR's, and all manifolds are orientable, connected and boundaryless. A manifold M is said to be *closed* if M is compact and boundaryless.

If a proper map $p : M \rightarrow B$ is an approximate fibration, not only are the point inverses homotopy equivalent but also there exists an exact

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homotopy sequence between M , B and fibers of p as follows;

$$\cdots \rightarrow \pi_{i+1}(B) \rightarrow \pi_i(p^{-1}b) \rightarrow \pi_i(M) \rightarrow \pi(B) \rightarrow \cdots$$

It is very essential to examine whether a given decomposition map is an approximate fibration, for then, above exact homotopy sequence provides us structural informations about any one object by means of their interrelations with the rests. The aim of this paper is to suggest a solution to the question that when is a decomposition map an approximate fibration.

A decomposition G of a metric space X is said to be *upper semicontinuous* if each $g \in G$ is compact in X and if, for each open subset U of X containing g , there exists another open subset V of X containing g such that every $g' \in G$ intersecting V is contained in U . Upper semicontinuous decomposition G makes a decomposition map proper and guarantees metrizability of the decomposition space M/G .

A closed n -manifold N is called a *codimension k fibrator* if whenever there is a usc decomposition G of an arbitrary $(n+k)$ -manifold M such that each element of G is homotopy equivalent to N and $\dim M/G < \infty$, then $p: M \rightarrow M/G$ is an approximate fibration.

Liem [14] proved that S^n ($n \geq 2$) is a codimension 1 fibrator, and Daverman [4] showed that if G is a decomposition of an $(n+1)$ -manifold M into continua having the shape of arbitrary closed n -manifolds then M/G is a 1-dimensional manifold, furthermore, if each element of G is locally flat in M , then p is an approximate fibration.

Daverman and Walsh [8] [9] showed that M/G is a 2-manifold for such a upper semicontinuous decomposition G of $(n+2)$ -manifold M and investigated decompositions G of $(n+k)$ -manifold M inducing a generalized manifold M/G . Until now, known codimension 2 fibrators are as follows: all simply connected closed manifold N^n , projective n -spaces P^n , all closed surfaces N^2 with nonzero Euler characteristic [7], all hyperbolic, $SL_2(R)$, Nil 3-manifolds, and all connected sum of nonsimply connected 3-manifolds except $P^3 \# P^3 [6]$, a finite product of closed orientable surfaces with negative Euler characteristics [13] and the others. We are going to add a few spaces to this codimension 2 fibrator's list.

Consider a usc decomposition G of a manifold M into closed orientable manifolds. There are useful winding functions defined locally on $B(= M/G)$ in the following manner. Each $b \in B$ has neighborhoods $U_0 \subset U$ such that $p^{-1}(U)$ retracts to $p^{-1}(b)$ and $p^{-1}(U_0)$ deformation retracts to $p^{-1}(b)$ in $p^{-1}(U)$, then the inclusion induced

$$\Psi_b : H_n(p^{-1}b) \rightarrow H_n(p^{-1}U)$$

is an isomorphism onto the image of

$$\Psi : H_n(p^{-1}U_0) \rightarrow H_n(p^{-1}U).$$

For any $c \in U_0$, the image of

$$\Psi_c : H_n(p^{-1}c) \rightarrow H_n(p^{-1}U)$$

is contained in $im\Psi$. Hence

$$\Psi_b^{-1} \circ \Psi_c : H_n(p^{-1}c) \rightarrow H_n(p^{-1}b)$$

is a well-defined homomorphism between two copies of \mathbb{Z} , meaning that up to sign it amounts to multiplication by some integer $q_c \geq 0$. The *local winding function* $\alpha_b : U_0 \rightarrow \mathbb{Z}$ at b is determined by the rule $\alpha_b(c) = q_c$. The *continuity set* C of $p : M \rightarrow B$ consists of those points $b \in B$ such that α_b is continuous in some neighborhood of b . The continuity set C is a dense, open subset of B [3], and its complement in B is locally finite for the case $k = 2$ [8]. It is worth while to research when the complement of C in B is locally finite for general case k .

Coram and Duvall [2] gave several characterizations for an approximate fibration. One of them is that a proper map $p : M \rightarrow B$ is an approximate fibration if and only if it is k -movable for all k (for details, see [2]), since then, this criterion has been the most used to check under which conditions inverse images of p are homotopy equivalent. The following terms help a lot for looking into these conditions.

A closed manifold N is called *Hopfian* if every degree one map $N \rightarrow N$ which induces a π_1 -isomorphism is a homotopy equivalence. A group H is *Hopfian* if every epimorphism $\Theta : H \rightarrow H$ is necessarily an isomorphism, while a finitely presented group H is *hyperhopfian* if

every homomorphism $\Psi : H \rightarrow H$ with $\Psi(H)$ normal and $H/\Psi(H)$ cyclic is an automorphism.

The symbol χ is used to denote *Euler characteristic*.

A group H is said to be *residually finite* if for each $e_H \neq h \in H$, there exists a finite group A and a homomorphism $\Phi : H \rightarrow A$ with $\Phi(h) \neq e_A$.

2. Preliminaries

In the study of a decomposition map $p : M^{n+k} \rightarrow B$ from $(n+k)$ -manifold M^{n+k} , codimension 2 is much more advantageous and accessible than other dimensions on account of following result.

THEOREM 2.1 [8]. *If G is a usc decomposition of an $(n+2)$ -manifold M into closed n -manifolds, then the decomposition space $B(= M/G)$ is a 2-manifold and $D = B - C$ is locally finite in B , where C represents the continuity set of $p : M \rightarrow B$.*

Since D is locally finite for $k = 2$, we can assume without loss of generality that B is an open disk and p is an approximate fibration over $B - b$ for some $b \in B$, provided p is an approximate fibration over the continuity set. In such a case, the homotopy exact sequence for p over $B - b$ can be reduced to a short exact sequence because of the simply connected property of $B - b$ and hence the hyperhopfian property of $\pi_1(N)$ displays its powerful ability to extend the continuity set C to B .

In [5], Daverman proved the following theorems on the codimension 2 case.

THEOREM 2.2. *Every $(k-1)$ -connected closed manifold $N^n (k > 0)$ is a codimension k fibrator.*

COROLLARY 2.3. *For $n \geq 1$ and $n \geq k$, S^n is a codimension k fibrator.*

THEOREM 2.4. *Every closed surface F for which $\chi(F) \neq 0$ is a codimension 2 fibrator.*

The problem whether or not a product $N_1 \times N_2$ of codimension 2 fibrators is a codimension 2 fibrator is not yet completely settled. As a

partial solution for this and an extension of Theorem 2.4, Im obtained the following consequence.

THEOREM 2.5 [13]. *Any finite product $N = F_1 \times F_2 \times \cdots \times F_m$ of closed surfaces F_i ($i = 1, \dots, m$) of genus at least 2 is a codimension 2 fibrator.*

In this paper, we extend Im's result to the extent that a product $S^n \times N$ is a codimension 2 fibrator, where S^n is an n -sphere and N is given a finite product $F_1 \times \cdots \times F_m$ of closed surfaces F_i ($i = 1, \dots, m$) of genus at least 2.

Consider a Hopfian manifold N^n with hyperhopfian fundamental group. Hopfian manifold N^n makes any degree one map $N \rightarrow N$ a homotopy equivalence, so that the decomposition map is an approximate fibration on the continuity set [5]. In the case $k = 2$, hyperhopfian property of the fundamental group $\pi_1(N)$ implies that the continuity set is the whole base space [7], and thus one can easily deduce the following theorem.

THEOREM 2.6 [7]. *All closed, Hopfian manifolds with hyperhopfian fundamental group are codimension 2 fibrators.*

According to this Theorem, Theorem 2.4 can be obtained as its immediate corollary, for the fundamental group of any closed surface F with $\chi(F) \neq 0$ is hyperhopfian and F is aspherical, implying that F is a Hopfian manifold.

Let us say that a decomposition G of a manifold into ANR's has Property $R \cong$ if, for each $g_0 \in G$, a retraction $R : U \rightarrow g_0$ defined on some open set $U \supset g_0$ induces π_1 -isomorphisms $(R|g)_\# : \pi_1(g) \rightarrow \pi_1(g_0)$ for all g sufficiently close to g_0 . Similarly, we say that G has Property $R_* \cong$ if $R : U \rightarrow g_0$ as above restricts to H_1 -isomorphisms $(R|g)_* : H_1(g) \rightarrow H_1(g_0)$ for all g sufficiently close to g_0 . If a decomposition G has Property $R_* \cong$ and $g_0 \in G$, then a given neighborhood retraction $R : U \rightarrow g_0$ restricts to a degree one map $g \rightarrow g_0$. Substantially, this fact exhibits a very efficient method to determine if a Hopfian manifold is a codimension k fibrator.

PROPOSITION 2.8 [7]. *Suppose N^n is a Hopfian manifold and G is a usc decomposition of $(n + 2)$ -manifold M with Property $R \cong$ such that*

each element of G is homotopy equivalent to N^n and $\dim M/G < \infty$. Then $p : M \rightarrow B$ is an approximate fibration.

PROPOSITION 2.9 [12]. *If H is a finitely generated, residually finite group, then H is a Hopfian group.*

It is known that fundamental groups of surfaces are residually finite [11] and thus it follows easily from the above Proposition that every closed surface has a Hopfian fundamental group.

3. Main results

In this section, we are going to prove two theorems. First, we verify that the product $S^n \times F$ of S^n and any closed surface F with $\chi(F) < 0$ is a codimension 2 fibrator, and then show that it holds for any finite product $F_1 \times \cdots \times F_m$ of closed surfaces with $\chi(F_i) < 0$ instead of a closed surface F .

Henceforward, let us put M a $(n+2)$ -manifold and G a upper semi-continuous decomposition of M into copies of a closed manifold N^n .

THEOREM 3.1. *A product $N^{n+2} = S^n \times F$ of n -sphere S^n ($n \geq 2$) and a closed orientable surface F with $\chi(F) < 0$ is a codimension 2 fibrator.*

Proof. Since $\pi_1(S^n \times F) \cong \pi_1(S^n) \oplus \pi_1(F) \cong \pi_1(F)$ and it is hyperhopfian, it suffices that we show that $N = S^n \times F$ is a Hopfian manifold by means of Theorem 2.6. Let $R : N(= S^n \times F) \rightarrow N(= S^n \times F)$ be an arbitrary degree one map. Then it induces a fundamental group epimorphism [12] and actually $R_{\#} : \pi_1(S^n \times F) \rightarrow \pi_1(S^n \times F)$ is an isomorphism owing to the hyperhopfian property of $\pi_1(N)$.

Consider $R_{\#} : \pi_i(S^n \times F) \rightarrow \pi_i(S^n \times F)$ for $2 \leq i \leq n-1$. In this case, the i th homotopy group $\pi_i(S^n \times F)$ of $S^n \times F$ is trivial and thus $R_{\#}$ is the trivial π_i -isomorphism.

Next, to show that R induces a π_n -isomorphism, let $R_{\#} : \pi_n(S^n \times F) \rightarrow \pi_n(S^n \times F)$ be an endomorphism of the n th homotopy group of $S^n \times F$. By the Poincaré duality, degree one map R induces a homology group isomorphism $R_{\star} : H_n(S^n \times F) \rightarrow H_n(S^n \times F)$. Consider the

following commutative diagram:

$$\begin{array}{ccc}
 \pi_n(S^n \times F) & \longrightarrow & \pi_n(S^n \times F) \\
 i_{\#} \uparrow & & \downarrow pr_{\#} \\
 \pi_n(S^n) & \xrightarrow{(pr \circ R \circ i)_{\#}} & \pi_n(S^n) \\
 \cong \downarrow & & \cong \downarrow \text{Whitehead theorem} \\
 H_n(S^n) & \xrightarrow{(pr \circ R \circ i)_{*}} & H_n(S^n) \\
 i_{*} \downarrow & & pr_{*} \uparrow \text{K\"unneth theorem} \\
 H_n(S^n \times F) & \xrightarrow{R_{*}} & H_n(S^n \times F)
 \end{array}$$

where $i : S^n \rightarrow S^n \times F$ is an inclusion and $pr : S^n \times F \rightarrow S^n$ is a projection.

K\"unneth theorem for homology implies that $H_n(S^n \times F)$ is isomorphic to $H_n(S^n)$, that is, $(pr \circ R \circ i)_{*} : H_n(S^n) \rightarrow H_n(S^n)$ is an isomorphism. Since n -sphere S^n is $(n-1)$ -connected, we can apply the Whitehead theorem and hence it follows that $(pr \circ R \circ i)_{\#} : \pi_n(S^n) \rightarrow \pi_n(S^n)$ is an isomorphism. From the fact that $\pi_n(S^n \times F) \cong \pi_n(S^n)$, the homomorphisms $i_{\#}$ and $p_{\#}$ are certified to be isomorphisms, and they imply that $R_{\#} : \pi_n(S^n \times F) \rightarrow \pi_n(S^n \times F)$ is an isomorphism.

The case $i \geq n + 1$ is in the same circumstances as the case $i = n$ and the desired result is obtained.

So far, we showed that $R_{\#}$ is π_i -isomorphism for each case i by case, this means that a degree one map R induces a homotopy equivalence. Thus N is a Hopfian manifold.

From now on, we show that Theorem 3.1 can be extended to the finite product $F_1 \times \cdots \times F_m$ of closed surfaces F_i with $\chi(F_i) < 0$ instead of a closed surface F . Recall that every closed surface has a residually finite fundamental group. From the definition of residually finite group, it is easily derived that the fundamental group of $N = S^n \times F_1 \times \cdots \times F_m$ is residually finite. Since $\pi_1(N)$ is finitely generated, we can apply Proposition 2.9 and obtain the conclusion that the fundamental group of a finite product $N = S^n \times F_1 \times \cdots \times F_m$ is Hopfian.

THEOREM 3.2. *A finite product $N = S^n \times F_1 \times \cdots \times F_m$ of n -sphere S^n and closed orientable surfaces $F_i (i = 1, \dots, m)$ with $\chi(F_i) < 0$ is a codimension 2 fibration.*

Proof. We begin by proving that N is a Hopfian manifold. Suppose that $R : N \rightarrow N$ is a degree one map. Then R induces a fundamental group epimorphism and so $R_{\#} : \pi_1(N) \rightarrow \pi_1(N)$ is an isomorphism by the Hopfian property of $\pi_1(N)$. For $2 \leq i \leq n-1$, $\pi_i(N)$ is a trivial group, and then it is obvious that $R_{\#}$ is a π_i -isomorphism. In the case $i \geq n$, the π_i -isomorphic property of $R_{\#}$ can be extracted from the same argument as the proof of Theorem 3.1. Therefore, the degree one map R induces a homotopy equivalence, this proves that N is a Hopfian manifold.

Now, we must show that the decomposition G has Property $R \cong$, for then p is an approximate fibration by Proposition 2.8. As a matter of fact, the proof of Main Theorem in [13] says that the decomposition satisfies the Property $R_{\star} \cong$, provided N is $F_1 \times \cdots \times F_m$. Since the first homology group of $F_1 \times \cdots \times F_m$ is the same as that of $S^n \times F_1 \times \cdots \times F_m$, the given decomposition satisfies the Property $R_{\star} \cong$, provided $N = S^n \times F_1 \times \cdots \times F_m$. Therefore, R is a degree one map [7, Lemma 5.2.] and as in the first part of this proof, the given decomposition satisfies Property $R \cong$.

COROLLARY 3.3. *Let M^{n+2} denote a simply connected $(n+2)$ -manifold and N^n a finite product of 2-sphere and closed surfaces with negative Euler characteristics. Then there is no usc decomposition of M^{n+2} into copies of N^n .*

Proof. Assume, on the contrary, that G is such a usc decomposition of M^{n+2} . Then B is a simply connected 2-manifold, so that it is either S^2 or R^2 . Hence $\pi_2(B)$ is \mathbb{Z} or $\{0\}$. Since N^n is a codimension 2 fibration, there is an exact homotopy sequence

$$\pi_2(B) \rightarrow \pi_1(N^n) \rightarrow \pi_1(M^{n+2}) \cong 1,$$

which leads to the impossibility that $\pi_1(N^n)$ is cyclic.

COROLLARY 3.4. *If G is a usc decomposition of M^{n+2} into copies of a finite product of 2-sphere and closed surfaces with negative Euler*

characteristic and $\chi(B) \leq 0$, then M^{n+2} is aspherical and $\pi_1(M^{n+2})$ is an extension of $\pi_1(N^n)$ by $\pi_1(B)$.

Proof. For $k \geq 2$ part of the exact homotopy sequence of the approximate fibration $p : M^{n+2} \rightarrow B$ shows

$$1 \cong \pi_k(N^n) \rightarrow \pi_k(M^{n+2}) \rightarrow \pi_k(B) \cong 1,$$

and implies that M^{n+2} is aspherical. The same sequence and the fact that $\pi_2(B) \cong 1$ expose the group extension.

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