J. Korean Math. Soc. **33** (1996), No. 1, pp. 89–99

THE POINT SPECTRUM OF THE LINEARIZED BOLTZMANN OPERATOR WITH THE EXTERNAL–POTENTIAL TERM IN AN EXTERIOR DOMAIN

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1. Introduction

The nonlinear Boltzmann equation with an external-force potential $\phi = \phi(x)$ has the form,

$$\frac{\partial f}{\partial t} + \Lambda f = Q(f, f). \tag{1.1}$$

This equation describes the time evolution of rarefied gas acted upon by the external force $\mathbb{F} = -\nabla\phi$. $f = f(t, x, \xi)$ is the unknown function denoting the density of gas particles at time $t \geq 0$, at a point $x \in \Omega$, and with a velocity $\xi \in \mathbb{R}^3$. Ω is a domain $\subseteq \mathbb{R}^3$ in which the rarefied gas is confined. Λ and $Q(\cdot,\cdot)$ are the following operators (see [1-2]):

$$\begin{aligned} \Lambda &\equiv \xi\cdot\nabla_x - \nabla_x\phi\cdot\nabla_\xi, \\ Q(g,h) &\equiv \frac{1}{2}\int_{\xi'\in\mathbb{R}^3, s\in S^2} B(\theta, |\xi-\xi'|) \\ &\quad \times \{g(\eta)h(\eta') + g(\eta')h(\eta) - g(\xi)h(\xi') - g(\xi')h(\xi)\}d\xi' ds, \end{aligned}$$

where $g(\eta) = g(t, x, \eta)$, etc., $\eta = \xi - ((\xi-\xi')\cdot s)s$, $\eta' = \xi' + ((\xi-\xi')\cdot s)s$, and $\cos\theta = \frac{(\xi-\xi')\cdot s}{|\xi-\xi'|}$, $s \in S^2$. S^2 denotes the unit sphere whose center is the origin. $B(\theta, V)$ is a nonnegative known function of $(\theta, V) \in [0,\pi]\times[0,+\infty)$. We will impose the following (see [1-2]):

Received November 5, 1994.

1991 AMS Subject Classification: 76P05, 45K05.

Key words and phrases: point spectrum, linearized Boltzmann operator, external potential, exterior domain.

ASSUMPTION 1.1. $\frac{B(\theta,V)}{|\sin\theta\cos\theta|} \leq c_{1.1}(V + V^{\varepsilon-1})$, *where* $c_{1.1} > 0$ *and* $0 < \varepsilon < 1$ *are constants independent of* (θ, V).

Under this assumption we linearize (1.1) around the absolute Maxwellian state $M \equiv \exp(-E(x,\xi))$, where $E(x,\xi) \equiv \phi(x) + \frac{|\xi|^2}{2}$. Substituting $f = M + M^{\frac{1}{2}}u$ in (1.1), and dropping the nonlinear term, we obtain the linearized Boltzmann equation,

$$\frac{\partial u}{\partial t} = Bu, \tag{1.2}$$

where $B \equiv A + e^{-\phi(x)}K$, and $A \equiv -\Lambda + e^{-\phi(x)}(-\nu)$. The operator B is the linearized Boltzmann operator. $\nu = \nu(\xi)$ is a multiplication operator, and K is an integration operator with a symmetric kernel. ν and K act on ξ only. These operators satisfy the following (see [1-2]):

LEMMA 1.2. (i) *There exists a positive constant* $c_{1.2}$ *such that for any* $\xi \in \mathbb{R}^3$, $0 < \nu(\xi) \leq c_{1.2}(1 + |\xi|)$.

(ii) K *is a self-adjoint compact operator on* $L^2(\mathbb{R}^3_\xi)$.

(iii) $(-\nu + K)$ *is a self-adjoint nonpositive operator on* $L^2(\mathbb{R}^3_\xi)$.

(iv) *The point spectrum of* $(-\nu + K)$ *contains 0, and the null space is spanned by* $\xi_j \exp\left(-\frac{|\xi|^2}{4}\right)$, $j = 1,2,3$, $\exp\left(-\frac{|\xi|^2}{4}\right)$, *and* $|\xi|^2 \exp\left(-\frac{|\xi|^2}{4}\right)$, *where* ξ_j *is the* j*-th component of* ξ, $j = 1,2,3$, *i.e.*, $\xi = (\xi_1, \xi_2, \xi_3)$.

It is important to investigate decaying of solutions of (1.2) (see [3, p. 768], [4, p. 241], and [5, p. 1827]). For this purpose we need to first inspect the point spectrum of B on the imaginary axis and the corresponding eigenspaces. Because we can obtain estimates for the decaying of solutions of (1.2) only in function spaces perpendicular to the eigenspaces corresponding to eigenvalues of B on the imaginary axis (cf. [1-2]).

In [6] we have already investigated this subject when $\Omega = \mathbb{R}^3$, and by making use of the result in [6], we have obtained decay estimates for solutions of (1.2) (cf. [3-5]). In the present paper, we will study that subject when Ω is an exterior domain, i.e., when $\mathbb{R}^3 \setminus (\Omega \cup \partial\Omega)$ is a bounded domain. The main result is Theorem 4.1. The boundary condition considered is the perfectly reflective boundary condition. We assume that the boundary $\partial\Omega$ is sufficiently smooth, and that the traces upon $\partial\Omega$ of functions contained in the domain of B are square-integrable with respect to some measure on $\partial\Omega \times \mathbb{R}^3$.

In [6], the eigenvalues of B and the corresponding eigenfunctions have only to satisfy the following:

$$\mu v = Bv. \tag{1.3}$$

In this paper, we obtain μ and v which satisfy (1.3), and moreover we need to examine whether v satisfies the perfectly reflective boundary condition or not. The forms of eigenfunctions of B are heavily restricted by this fact, and hence we have to perform more complicated calculations than those in [6].

However, for the same reason, some eigenvalues of B in [6] are not eigenvalues of B in the present paper. As a result, the structure of the point spectrum is simplified; the point spectrum is only equal to $\{0\}$ in the present paper.

This paper consists of 4 sections. §2 presents preliminaries. In §3, we obtain necessary conditions for the point spectrum of B and the corresponding eigenspaces. In §4 we prove the main theorem.

RMARK 1.3. We can also investigate, by the method developed in this paper, the case where Ω is bounded. We will study this subject in another paper.

ACKNOWLEDGEMENTS. The authors would like to express their deepest gratitude to Professor Shin-ichi Nakagiri, Kobe University, for his helpful advice. This work was supported by Grant-in-aid for Scientific Research 07640209, Ministry of Education of Japan.

2. Preliminaries

We impose the following on the domain Ω and the external-force potential $\phi = \phi(x)$:

ASSUMPTION 2.1. (i) *$\mathbb{R}^3 \setminus (\Omega \cup \partial\Omega)$ is a bounded domain.*
(ii) *$\partial\Omega$ is a sufficiently smooth surface.*

ASSUMPTION 2.2. (i) $\phi = \phi(x)$ *is sufficiently smooth and real-valued in* Ω, *and is continuous in* $\partial\Omega \cup \Omega$.

(ii) $L^2(\Omega)$ *contains* $e^{-\frac{\phi(x)}{2}}$, $\phi(x)e^{-\frac{\phi(x)}{2}}$, *and* $|x|e^{-\frac{\phi(x)}{2}}$.

(iii) *There exists a constant* $c_{2.2}$ *such that for any* $x \in \Omega$ $\phi(x) \geq c_{2.2}$.

REMARK 2.3. (i) Assumption 2.1,(ii), and Assumption 2.2,(i) are strong conditions. In fact, it is sufficient to assume, in place of them, that $\partial\Omega$ and $\phi = \phi(x)$ belong to the C^2-class. However, to fully argue conditions on the regularity of $\partial\Omega$ and $\phi = \phi(x)$ would carry us far away from the main subject in this paper. Hence we accept them for simplicity.

(ii) Assumption 2.2,(ii) will be discussed in §4.

We define $S_j \equiv \{(x,\xi) \in \partial\Omega \times \mathbb{R}^3;\ (-1)^j n(x)\cdot\xi < 0\}$, $j = 1,2$, where $n = n(x)$ denotes the outer unit normal of $\partial\Omega$ at $x \in \partial\Omega$.

We consider our problem in the complex Hilbert space $L^2(\Omega_x \times \mathbb{R}^3_\xi)$. By $L^2(S_j;\rho)$, we denote the space of square-integrable functions of $(x,\xi) \in S_j$ with respect to $\rho(x,\xi)d\sigma_x d\xi$, $j = 1,2$, where $\rho = \rho(x,\xi) \equiv |n(x)\cdot\xi|$. $d\sigma_x$ denotes an infinitesimal surface element of $\partial\Omega_x$.

By $D(L)$ we denote the domain of an operator L. We define $D(\Lambda) \equiv \{v = v(x,\xi) \in L^2(\Omega_x \times \mathbb{R}^3_\xi);\ \Lambda v \in L^2(\Omega_x \times \mathbb{R}^3_\xi)$, and $v = v(x,\xi)$ satisfies the following boundary conditions:

$$\text{(SI)} \qquad (\gamma_j v(\cdot,\cdot))(x,\xi) \in L^2(S_j;\rho), \quad j = 1,2,$$

$$\text{(PRBC)} \quad (\gamma_1 v(\cdot,\cdot))(x,\xi) = (\gamma_2 v(\cdot,\cdot))(x,\xi - 2(n(x)\cdot\xi)n(x)),$$

for any $(x,\xi) \in S_1\}$. γ_j, $j = 1,2$, denote the trace operators along the characteristic curves, which are defined by the following system of ordinary differential equations:

$$\frac{dx}{dt} = \xi, \quad \frac{d\xi}{dt} = -\nabla\phi(x). \tag{2.1}$$

γ_j, $j = 1,2$, make functions defined in $\Omega_x \times \mathbb{R}^3_\xi$ correspond to those defined in S_j, $j = 1,2$, respectively.

We similarly define $D(A) \equiv \{v = v(x,\xi) \in L^2(\Omega_x \times \mathbb{R}^3_\xi);\ Av \in L^2(\Omega_x \times \mathbb{R}^3_\xi)$, and $v = v(x,\xi)$ satisfies (SI) and (PRBC)$\}$. It follows from Assumption 2.2,(iii) and Lemma 1.2,(ii) that $e^{-\phi}K$ is a bounded operator in $L^2(\Omega_x \times \mathbb{R}^3_\xi)$. In virtue of this fact, we can define $D(B) \equiv D(A)$.

By $a(\phi)$ ($a(\Omega)$, respectively) we denote the set of all axes of symmetry of $\phi = \phi(x)$ (Ω, respectively).

REMARK 2.4. (i) If v, $\xi \cdot \nabla_x v \in L^2(\Omega_x \times \mathbb{R}^3_\xi)$, then $v = v(x,\xi)$ is absolutely continuous along the characteristic lines of $\xi \cdot \nabla_x$. We can construct the trace operators along the characteristic lines of $\xi \cdot \nabla_x$. See [7, Chapter 2]. Performing calculations similar to those in obtaining these facts, we can deduce that if

$$v, \Lambda v \in L^2(\Omega_x \times \mathbb{R}^3_\xi), \tag{2.2}$$

then $v = v(x,\xi)$ is absolutely continuous along the characteristic curves of Λ. We can construct the trace operators γ_j, $j = 1, 2$. In addition, combining (SI) and (PRBC), and performing calculations similar to those in [7, Chapter 2], we see that if $v \in D(\Lambda)$, then

$$(v, \Lambda v) + (\Lambda v, v) = I_1(v) - I_2(v) = 0, \tag{2.3}$$

where the brackets denote the inner product in $L^2(\Omega_x \times \mathbb{R}^3_\xi)$, and

$$I_j(v) \equiv \int_{S_j} v(x,\xi)\overline{v(x,\xi)}\rho(x,\xi)d\sigma_x d\xi, \quad j = 1, 2.$$

(2.3) will play an important role in the next section.

(ii) By imposing (SI), we heavily restrict the domains of the operators. However, we immediately find it nearly impossible to obtain (SI) from only (2.2), without imposing additional assumptions such as the convexity of $\mathbb{R}^3 \setminus (\Omega \cup \partial\Omega)$. Moreover it is very difficult to obtain (2.3) from only (PRBC) without (SI), because there is a possibility that $I_j(v) = +\infty$, $j = 1, 2$. For these reasons, we will accept (SI).

3. Necessary Conditions

Let us obtain necessary conditions for μ and $v \in D(B)$ to satisfy (1.3).

LEMMA 3.1. *Suppose that $v = v(x,\xi) \in D(B)$ is not identically equal to 0, and that $Re\ \mu \geq 0$. If μ and v satisfy (1.3), then*

$$\mu = 0, \tag{3.1}$$

$$\Lambda v = 0, \tag{3.2}$$

and v has the form,

$$v = \left(\sum_{j=1}^{3} a_j \xi_j + a_4 |\xi|^2 + a_5 \right) \exp\left(-\frac{E(x,\xi)}{2} \right), \tag{3.3}$$

where $E(x,\xi) \equiv \phi(x) + \frac{|\xi|^2}{2}$. The coefficients $a_j = a_j(x)$, $j = 1, \ldots, 5$, are complex-valued functions of $x \in \Omega$ which satisfy the following (3.4-6):

$$a_j = \alpha_j + \sum_{k=1}^{3} \alpha_{jk} x_k, \quad j = 1,2,3, \tag{3.4}$$

$$a_4 \quad \text{is a complex constant,} \tag{3.5}$$

$$a_5 = 2a_4 \phi(x) + \beta_0, \tag{3.6}$$

where β_0 is a complex constant. The coefficients α_j, α_{jk}, $j,k = 1,2,3$, are complex constants which satisfy the following (3.7-8):

$$\alpha_{jk} + \alpha_{kj} = 0, \quad j,k = 1,2,3. \tag{3.7}$$

(3.8): *Define*

$$(\alpha, \beta) \equiv ((Re\ \alpha_1, Re\ \alpha_2, Re\ \alpha_3), (Re\ \alpha_{23}, Re\ \alpha_{31}, Re\ \alpha_{12})),$$
$$((Im\ \alpha_1, Im\ \alpha_2, Im\ \alpha_3), (Im\ \alpha_{23}, Im\ \alpha_{31}, Im\ \alpha_{12})).$$

If $a(\phi) \cap a(\Omega)$ is empty, then $(\alpha,\beta) = (0,0)$. If $\phi = \phi(x)$ and Ω have only one common axis of symmetry, i.e., if $a(\phi) \cap a(\Omega) = \{\ell\}$, then (α,β) satisfies $\beta // \ell$ and $\alpha = -\gamma \times \beta$ for any $\gamma \in \ell$. If both $\phi = \phi(x)$ and Ω are spherically symmetric with respect to a point $\gamma \in \mathbb{R}^3$, then (α,β) satisfies $\alpha = -\gamma \times \beta$.

REMARK 3.2. We easily see that if $a(\phi) \cap a(\Omega)$ is not empty, then only the following two cases exist: (1) $\phi = \phi(x)$ and Ω have only one common axis of symmetry. (2) $\phi = \phi(x)$ and Ω are spherically symmetric with respect to only one point.

Proof of Lemma 3.1. Let us prove (3.1-3) and (3.5). Calculate the L^2-inner products of v and both sides of (1.3), and take their real parts. Recalling that Re $\mu \geq 0$, and applying (2.3) and Lemma 1.2, we obtain (3.3) and the following:

$$\text{Re } \mu = 0, \tag{3.9}$$

$$\mu v = -\Lambda v. \tag{3.10}$$

Substituting (3.3) in (3.10), and comparing the coefficients of $\xi_j, \xi_j\xi_k$, $\xi_j|\xi|^2$, $j, k = 1, 2, 3$, we obtain (3.5) and the following (cf. [6, p. 187]):

$$\mu a_5 - \sum_{j=1}^{3} a_j \frac{\partial \phi}{\partial x_j} = 0, \tag{3.11}$$

$$\mu a_j + \frac{\partial a_5}{\partial x_j} - 2a_4 \frac{\partial \phi}{\partial x_j} = 0, \quad j = 1, 2, 3, \tag{3.12}$$

$$\frac{\partial a_j}{\partial x_k} + \frac{\partial a_k}{\partial x_j} = 0, \quad j \neq k, \quad j, k = 1, 2, 3, \tag{3.13}$$

$$\mu a_4 + \frac{\partial a_j}{\partial x_j} = 0, \quad j = 1, 2, 3, \tag{3.14}$$

where the derivatives are those in the sense of distribution. (3.5) and (3.11-14) are necessary conditions for (3.3) to satisfy (1.3).

By substituting (3.3) in (PRBC), we obtain the following necessary condition for (3.3) to satisfy (PRBC):

$$\nabla \psi \cdot a = 0, \quad \text{in } \partial\Omega, \tag{3.15}$$

where $a \equiv (a_j)_{j=1,2,3}$. $\psi = \psi(x)$ is a real-valued function of $x \in \mathbb{R}^3$ representing $\partial\Omega$ in such a way that $\partial\Omega = \{x \in \mathbb{R}^3; \psi(x) = 0\}$. The existence of $\psi = \psi(x)$ follows from Assumption 2.1 immediately.

Let us prove (3.1) by contradiction. Assume that $\mu \neq 0$. (3.5) and (3.12) give

$$\frac{\partial a_j}{\partial x_k} - \frac{\partial a_k}{\partial x_j} = 0, \quad j,k = 1,2,3.$$

It follows from these equalities and (3.13) that

$$\frac{\partial a_j}{\partial x_k} = 0, \quad j \neq k, \quad j,k = 1,2,3.$$

These equalities and (3.14) give

$$a_j = -\mu a_4 x_j + \beta_j, \quad j = 1,2,3, \tag{3.16}$$

where β_j, $j = 1,2,3$, are complex constants. Let $a_4 = 0$. Substituting (3.16) with $a_4 = 0$ in (3.15), and solving the equation thus obtained with respect to $\psi = \psi(x)$, we see that $\partial\Omega$ is an unbounded cylindrical surface. This is contradictory to Assumption 2.1,(i). Let $a_4 \neq 0$. Substituting (3.16) with $a_4 \neq 0$ in (3.15), and solving the equation thus obtained with respect to $\psi = \psi(x)$, we conclude that $\partial\Omega$ is an unbounded conical surface. This is contradictory to Assumption 2.1. Hence we obtain (3.1). (3.2) follows from (3.1) and (3.10) immediately.

Let us prove (3.4) and (3.6-7). Write (3.k.0) as (3.k) with $\mu = 0$, $k = 11, 12, 14$. (3.5) and (3.12.0) give (3.6). From (3.13) and (3.14.0) we have

$$\frac{\partial^2 a_j}{\partial x_k^2} = 0, \quad j,k = 1,2,3. \tag{3.17}$$

Combining (3.17) and (3.14.0), we deduce that a_j, $j = 1,2,3$, have the following forms:

$$a_j = \alpha_j + \alpha_{jk} x_k + \alpha_{j\ell} x_\ell + \gamma_j x_k x_\ell, \quad \{j,k,\ell\} = \{1,2,3\}, \tag{3.18}$$

where $\alpha_j, \alpha_{jk}, \alpha_{j\ell}$, and γ_j are complex constants. Substituting (3.18) in (3.13), and comparing the coefficients of x_j, $j = 1,2,3$, we obtain (3.4) and (3.7).

Let us prove (3.8). Substituting (3.4) with (3.7) in (3.11.0) and in (3.15), and noting that $\phi = \phi(x)$ and $\psi = \psi(x)$ are real-valued, we conclude that $\phi = \phi(x)$ and $\psi = \psi(x)$ satisfy equations of the same form,

$$\nabla\phi \cdot (\alpha + x \times \beta) = 0, \tag{3.19}$$

$$\nabla\psi \cdot (\alpha + x \times \beta) = 0, \tag{3.20}$$

where (α, β) is that in (3.8). Let $\beta = 0$ in (3.20). Suppose that $\alpha \neq 0$. Then, $\psi = \psi(x)$ is constant on any lines parallel to α. This fact and Assumption 2.1, (i), lead us to a contradiction. Hence, we have $\alpha = 0$. However, $(\alpha, \beta) = (0, 0)$ satisfies (3.8).

Let $\beta \neq 0$ in (3.20). Suppose that α is not perpendicular to β. Then, α is decomposed as follows: $\alpha = \alpha_0 + \alpha_\perp$, $\alpha_0 \neq 0$, $\alpha_0 // \beta$, $\alpha_\perp \perp \beta$. Since there exists a γ such that

$$\alpha_\perp = -\gamma \times \beta, \tag{3.21}$$

(3.20) can be rewritten as follows:

$$\nabla\psi \cdot (\alpha_0 + (x - \gamma) \times \beta) = 0. \tag{3.22}$$

The characteristic curves of this equation are helixes. In addition, those helixes have a unique common axis which is parallel to β and passes through γ. $\psi = \psi(x)$ is constant on those characteristic curves. However, this fact and Assumption 2.1,(i), lead us to a contradiction. Hence, $\alpha \perp \beta$, i.e., $\alpha = \alpha_\perp$. Therefore, (3.21) gives

$$\alpha = -\gamma \times \beta. \tag{3.23}$$

Substituting (3.23) in (3.19-20), we have

$$\nabla\phi \cdot ((x - \gamma) \times \beta) = 0, \quad \nabla\psi \cdot ((x - \gamma) \times \beta) = 0. \tag{3.24}$$

It follows from (3.24) that if $\beta \neq 0$, then both $\phi = \phi(x)$ and $\psi = \psi(x)$ are symmetric with respect to a line which is parallel to β and passes through γ. Making use of this fact and (3.23), and recalling Remark 3.2, we can obtain (3.8).

4. The Main Theorem

By σ_p we denote the point spectrum of B.

THEOREM 4.1. (i) $\sigma_p \cap \{\mu \in \mathbb{C}; Re\ \mu \geq 0\} = \{0\}$.

(ii) *If $a(\phi) \cap a(\Omega)$ is empty, then the null space of B is spanned by*

$$e^{-\frac{E(x,\xi)}{2}}, \quad E(x,\xi)e^{-\frac{E(x,\xi)}{2}}, \tag{4.1}$$

where $E(x,\xi) \equiv \phi(x) + \frac{|\xi|^2}{2}$.

(iii) *If $\phi = \phi(x)$ and Ω have only one common axis of symmetry, i.e., if $a(\phi) \cap a(\Omega) = \{\ell\}$, then the null space of B is spanned by*

$$e^{-\frac{E(x,\xi)}{2}}, \quad E(x,\xi)e^{-\frac{E(x,\xi)}{2}}, \quad ((x-\gamma)\times\xi)^{\ell} e^{-\frac{E(x,\xi)}{2}}, \quad \gamma \in \ell, \tag{4.2}$$

where by $((x-\gamma)\times\xi)^{\ell}$ we denote the projection of $(x-\gamma)\times\xi$ upon the line ℓ.

(iv) *If $\phi = \phi(x)$ and Ω are spherically symmetric with respect to a point $\gamma \in \mathbb{R}^3$, then the null space of B is spanned by*

$$e^{-\frac{E(x,\xi)}{2}}, \quad E(x,\xi)e^{-\frac{E(x,\xi)}{2}}, \quad ((x-\gamma)\times\xi)_j e^{-\frac{E(x,\xi)}{2}}, \quad j = 1,2,3, \tag{4.3}$$

where by $((x-\gamma)\times\xi)_j$ we denote the j-th component of $(x-\gamma)\times\xi$, $j = 1,2,3$.

Proof. Write V as the set of all functions of the form (3.3) whose $a_j = a_j(x)$, $j = 1, \ldots, 5$, satisfy (3.4-8). Making use of Lemma 3.1, we see that $\sigma_p \cap \{\mu \in \mathbb{C}; \mathrm{Re}\ \mu \geq 0\} \subseteq \{0\}$ and that the null space is contained in V.

It follows from Assumption 2.2,(ii) that $V \subseteq L^2(\Omega \times \mathbb{R}^3)$. From Assumption 2.2,(i), and Assumption 2.1, we see that all elements of V satisfy (SI). Moreover, we easily deduce that if $v \in V$, then v satisfies (PRBC) and (1.3) with $\mu = 0$. Hence, we deduce that $0 \in \sigma_p$ and that V is contained in the null space of B.

It follows from (3.4-8) that if ϕ and Ω satisfy the conditions of (ii-iv) of the present theorem respectively, then V is spanned by (4.1-3) respectively. Hence, we obtain the theorem.

REMARK 4.2. (i) We note that the null space of B varies with the common axes of symmetry of the external-force potential $\phi = \phi(x)$ and the domain Ω. The existence of the eigenfunctions (4.1-3) is closely related to the law of conservation of energy, to that of mass, and to that of angular momentum around the common axes of symmetry of ϕ and Ω (cf. [2, p. 159]).

(ii) If we do not accept Assumption 2.2, (ii), then the null space of B vanishes or its dimension decreases. For example, if $(\sum_{j=1}^{3} \beta_j x_j + \beta_4 \phi(x) + \beta_5)e^{-\frac{\phi(x)}{2}}$ is not contained in $L^2(\Omega)$ for any $(\beta_1, \dots, \beta_5) \neq (0, \dots, 0)$, then B has no eigenvalues on $\{\mu \in \mathbb{C}; \mathrm{Re}\, \mu \geq 0\}$.

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