J. Korean Math. Soc. **33** (1996), No. 1, pp. 31–38

A STABILITY RESULT FOR DIRICHLET PROBLEM OF THE FIRST–ORDER HAMILTON–JACOBI EQUATIONS

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1. Introduction

It is well known that Dirichlet problem of the first-order Hamilton-Jacobi equations

$$\text{(H-J)} \qquad u(x) + H(\nabla u(x)) = f(x), \quad x \in \mathbb{R}^N,$$

does not have a classical solution even though the Hamiltonian H is smooth. Therefore it is worth for us to deal with non-smooth solutions if we want a solution of (H-J) which satisfy the equations almost everywhere. The theory of first-order partial differential equations of Hamilton-Jacobi type has substantially developed with the introduction by Crandall and Lions [1] of the class of viscosity solutions, which turns out to be the correct class of generalized solutions for such type of equations. They also showed the uniqueness of generalized solutions that satisfy a so-called"viscosity" condition. The book by Lions [6] and papers by Jensen and Souganidis [5] and Souganidis [7] provided a view of the scope of the references to much of the recent literature. Cauchy problem of Hamilton-Jacobi equations was studied by Crandall and Lions [2]. Hong [3], [4] showed some regularity results for Cauchy problem of Hamilton-Jacobi equations.

This paper is organized as follows. In chapter 2, we give the definition of viscosity solutions of (H-J) in several space dimensions. We also review both uniqueness and stability of the viscosity solutions.

In chapter 3, we prove the following theorem that is the main result of this paper.

Received August 12, 1994.
1991 AMS Subject Classification: 70H20.
Key words and phrases: Hamilton-Jacobi equations.
This research was supported by Hoseo University.

THEOREM 1.1. *Let bounded and continuous functions u and v be the viscosity solutions of*

$$u(x) + H_1(\nabla u(x)) = f(x), \quad x \in \mathbb{R}^N,$$

and

$$v(x) + H_2(\nabla v(x)) = g(x), \quad x \in \mathbb{R}^N,$$

respectively, where H_1 and H_2 are Lipschitz continuous. Then

$$\|u - v\|_{L^\infty(\mathbb{R}^N)} \leq \|f - g\|_{L^\infty(\mathbb{R}^N)} + \|H_1 - H_2\|_{L^\infty(\mathbb{R}^N)}.$$

This stability result gives us an error estimate if we approximate the viscosity solutions of (H-J).

2. Viscosity solutions of (H-J)

The general reference for this section is [6]. To repeat, one cannot in general find a classical solution of (H-J) on $\mathbb{R}$, while bounded Lipschitz continuous "generalized solutions" in the almost-everywhere sense exist but are not unique. For example,

$$u(x) + |u_x(x)| = 1, \quad x \in \mathbb{R},$$

has two solutions that satisfy the equation almost everywhere, namely, $u = 1$ and

$$u = \begin{cases} 1 - e^x, & \text{if } x \leq x_0, \\ 1 - e^{2x_0 - x}, & \text{otherwise,} \end{cases}$$

which satisfies the equation classically except on the lines $x = x_0$ for all $x_0 \in \mathbb{R}$. Moreover, if u and v are generalized solutions of (H-J), then so are $\min(u, v)$ and $\max(u, v)$. In fact, if the problem is nonlinear, one can expect infinitely many generalized solutions. Crandall and Lions [1] resolved the uniqueness problem by introducing a notion of viscosity.

DEFINITION 2.1. A *viscosity subsolution* (respectively, *supersolution*) of (H-J) with $H \in C(\mathbb{R}^N)$ is a bounded function $u \in C(\mathbb{R}^N)$ such that for every $\phi \in C^1(\mathbb{R}^N)$:

$$\text{(2.1.1)} \qquad \begin{array}{l} \text{If } x_0 \text{ is a local maximum point of } u - \phi \text{ on } \mathbb{R}^N \\ textthen\ u(x_0) + H(\nabla\phi(x_0)) \leq f(x_0). \end{array}$$

(respectively,

$$\text{(2.1.2)} \qquad \begin{array}{l} \text{If } x_0 \text{ is a local minimum point of } u - \phi \text{ on } \mathbb{R}^N, \\ \text{then } u(x_0) + H(\nabla\phi(x_0)) \geq f(x_0).) \end{array}$$

DEFINITION 2.2. A *viscosity solution* of (H-J) is a bounded function $u \in C(\mathbb{R}^N)$ for which both (2.1.1) and (2.1.2) hold (i.e. u is both a viscosity subsolution and a viscosity supersolution).

REMARK. If u is a bounded classical solution of (H-J), then it is a viscosity solution, and if u is a viscosity solution of (H-J), then $u(x_0) + H(\nabla u(x_0)) = f(x_0)$ at any point (x_0) where u is differentiable.

LEMMA 2.3. *Suppose that u and v are viscosity solutions of*

$$v(x) + H_2(\nabla v(x)) = g(x), \quad x \in \mathbb{R}^N,$$

and

$$w(x) + H_2(\nabla w(x)) = f(x), \quad x \in \mathbb{R}^N,$$

respectively. Then

$$\|w - v\|_{L^\infty(\mathbb{R}^N)} \leq \|f - g\|_{L^\infty(\mathbb{R}^N)}.$$

Proof. See [1] and [6].

3. Stability of two viscosity solutions

We prove that viscosity solutions are stable under changes in the nonlinear Hamiltonians H as well as changes in the functions f. To prove that, we prepare two lemmas.

LEMMA 3.1. *Let u and w be the viscosity solutions of*

$$u(x) + H_1(\nabla u(x)) = f(x), \quad x \in \mathbb{R}^N,$$

and

$$w(x) + H_2(\nabla w(x)) = f(x), \quad x \in \mathbb{R}^N,$$

respectively, where H_1 and H_2 are Lipschitz continuous. Then

$$\|u - w\|_{L^\infty(\mathbb{R}^N)} \le \|H_1 - H_2\|_{L^\infty(\mathbb{R}^N)}.$$

Theorem 1.1 follows by combining Lemma 2.3 and Lemma 3.1.

LEMMA 3.2. *Assume that u and w are in Lemma 3.1. Let $\eta(z)$ be a smooth nonnegative function on $\mathbb{R}$ such that $\eta(-z) = \eta(z)$, $0 \le \eta(z) \le 1$, $\eta(0) = 1$ and $\eta(z) = 0$ if $|z| > 1$, and let $M = \max\{\|u\|_{L^\infty(\mathbb{R}^N)}, \|w\|_{L^\infty(\mathbb{R}^N)}\}$. Suppose that*

$$\sigma := \sup_{\mathbb{R}^N}(u(x) - w(x)) > 0.$$

For any $\epsilon > 0$, define

$$\psi(x, y) = u(x) - w(y) + (3M + \frac{\sigma}{2})\beta_\epsilon(x - y),$$

where $\beta_\epsilon(x)$ is defined on $\mathbb{R}^N$ by $\beta_\epsilon(x) = \prod_{i=1}^N \eta(\frac{x_i}{\epsilon})$. If

$$\sup_{|x|\ge R} |u(x)| \quad \text{and} \quad \sup_{|x|\ge R} |w(x)| \to 0 \quad \text{as } R \to \infty, \tag{3.2.1}$$

then there exists a point $(x_0, y_0) \in \mathbb{R}^N \times \mathbb{R}^N$ such that $\psi(x_0, y_0) \ge \psi(x, y)$ on $\mathbb{R}^N \times \mathbb{R}^N$.

Proof. Fix $\epsilon > 0$. If there is a sequence $\{(x_i, y_i)\}_{i\ge1}$ in $\mathbb{R}^N \times \mathbb{R}^N$ such that

$$\psi(x_i, y_i) \to \sup_{\mathbb{R}^N\times\mathbb{R}^N} \psi, \tag{3.2.2}$$

then (x_i, y_i) remains bounded by the following arguments.

First,

$$\begin{aligned}\sup_{\mathbb{R}^N\times\mathbb{R}^N}\psi &\geq u(x)-w(x)+(3M+\frac{\sigma}{2})\beta_\epsilon(x-x)\\ &= u(x)-w(x)+3M+\frac{\sigma}{2}\quad \text{for all } x\in\mathbb{R}^N.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{\mathbb{R}^N\times\mathbb{R}^N}\psi &\geq \sup_{\mathbb{R}^N}(u(x)-w(x))+3M+\frac{\sigma}{2}\\ &= \sigma+3M+\frac{\sigma}{2}\\ &= 3M+\frac{3}{2}\sigma.\end{aligned}\tag{3.2.3}$$

If $\beta_\epsilon(x-y)=0$, then

$$\begin{aligned}\psi(x,y) &= u(x)-w(y)\\ &\leq 2M.\end{aligned}$$

Hence, (3.2.2) implies that $\beta_\epsilon(x_i-y_i)>0$ for large i, whence $|x_i-y_i|<\epsilon$. If $|x_i|\to\infty$ and $|y_i|\to\infty$, then

$$\lim_{i\to\infty}\sup_{\mathbb{R}^N\times\mathbb{R}^N}\psi(x_i,y_i)\leq 3M+\frac{\sigma}{2}\quad\text{by 3.2.1.}$$

This contradicts (3.2.2) and (3.2.3). Therefore, $\{(x_i,y_i)\}_{i\geq 1}$ is a bounded sequence and there is a convergent subsequence of $\{(x_i,y_i)\}_{i\geq 1}$. Let (x_0,y_0) be the limit of the above subsequence. This completes the proof. □

Proof of Lemma 3.1. We will prove that σ, defined in Lemma 3.2, satisfies

$$\sigma\leq\|H_1-H_2\|_{L^\infty(\mathbb{R}^N)}.$$

By symmetry in u and w, we see that this implies

$$\|u-w\|_{L^\infty(\mathbb{R}^N)}\leq\|H_1-H_2\|_{L^\infty(\mathbb{R}^N)}.$$

We first assume that

$$\text{(A)} \qquad \sup_{|x|\geq R} |u(x)| \quad \text{and} \quad \sup_{|x|\geq R} |w(x)| \to 0 \quad \text{as } R \to \infty.$$

If $\sigma = 0$, then we are done. Otherwise, by Lemma 3.2, for any $\epsilon > 0$ we can find a point $(x_0, y_0) \in \mathbb{R}^N \times \mathbb{R}^N$ such that

$$u(x) - \Big(w(y_0) - (3M + \frac{\sigma}{2})\beta_\epsilon(x - y_0)\Big)$$

attains a local maximum at (x_0), whence

$$u(x_0) + H_1\Big(-(3M + \frac{\sigma}{2})\nabla_x\beta_\epsilon(x_0 - y_0)\Big) \leq f(x_0). \tag{3.1.1}$$

Similarly

$$-w(y) - \Big(-u(x_0) - (3M + \frac{\sigma}{2})\beta_\epsilon(x_0 - y)\Big)$$

attains local maximum at $y = y_0$ and therefore

$$w(y) - \Big(u(x_0) + (3M + \frac{\sigma}{2})\beta_\epsilon(x_0 - y)\Big)$$

attains local minimum at $y = y_0$. By the definition of the viscosity solution,

$$w(y_0) + H_2\Big((3M + \frac{\sigma}{2})\nabla_y\beta_\epsilon(x_0 - y_0)\Big) \geq f(y_0).$$

Since $\nabla_x\beta_\epsilon(x - y_0)|_{x=x_0} = -\nabla_y\beta_\epsilon(x_0 - y)|_{y=y_0}$,

$$w(y_0) + H_2\Big(-(3M + \frac{\sigma}{2})\nabla_x\beta_\epsilon(x_0 - y_0)\Big) \geq f(y_0). \tag{3.1.2}$$

Combining (3.1.1) and (3.1.2) gives
(3.1.3)

$$\begin{aligned} u(x_0) - w(y_0) \leq H_2\Big(-(3M + \frac{\sigma}{2})\nabla_x\beta_\epsilon(x_0 - y_0)\Big) \\ - H_1\Big((3M + \frac{\sigma}{2})\nabla_y\beta_\epsilon(x_0 - y_0)\Big) + f(x_0) - f(y_0). \end{aligned}$$

For all $x \in \mathbb{R}^N$,

$$
\begin{aligned}
u(x) - w(x) + 3M + \frac{\sigma}{2} &= \psi(x, x) \\
&\le \psi(x_0, y_0) \qquad (3.1.4)\\
&\le u(x_0) - w(y_0) + 3M + \frac{\sigma}{2}.
\end{aligned}
$$

Therefore, by 3.1.3 and 3.1.4,

$$
\begin{aligned}
\sigma \le & H_1\Big(-(3M + \frac{\sigma}{2})\nabla_x \beta_\epsilon(x_0 - y_0)\Big) - H_2\Big((3M + \frac{\sigma}{2})\nabla_y \beta_\epsilon(x_0 - y_0)\Big) \\
& + f(x_0) - f(y_0) \\
\le & \|H_1\Big(-(3M + \frac{\sigma}{2})\nabla_x \beta_\epsilon(x_0 - y_0)\Big) \\
& - H_2\Big(-(3M + \frac{\sigma}{2})\nabla_x \beta_\epsilon(x_0 - y_0)\Big)\|_{L^\infty(\mathbb{R}^N)} + \omega_f(\epsilon)
\end{aligned}
$$

where $\omega_f(\epsilon)$ is the modulus of continuity of f. Since ϵ is arbitrary,

$$
\sigma \le \|H_1 - H_2\|_{L^\infty(\mathbb{R}^N)}.
$$

We now drop the assumption (A). For $R > 0$, let $\rho(x)$ be a smooth function having support in the ball $B(0, R{+}1) = \{x \in \mathbb{R}^N \mid |x| \le R{+}1\}$ such that $\rho(x) = 1$ on $|x| \le R$. Suppose that $u^\rho(x) = \rho(x)u(x)$ and $w^\rho(x) = \rho(x)w(x)$. Then the corresponding viscosity solutions $u^\rho(x)$ and $w^\rho(x)$ have the following properties:

$$
\begin{aligned}
u^\rho(x) = u(x) \quad &\text{on} \quad |x| < R - |H_1|_{\mathrm{Lip}} \quad \text{and} \\
w^\rho(x) = w(x) \quad &\text{on} \quad |x| < R - |H_2|_{\mathrm{Lip}};
\end{aligned}
$$

see [6]. Let $L = \max\{|H_1|_{\mathrm{Lip}}, |H_2|_{\mathrm{Lip}}\}$. Then

$$
\begin{aligned}
\max_{|x|<R-L} |u(x) - w(x)| &= \max_{|x|<R-L} |u^\rho(x) - w^\rho(x)| \\
&\le \|H_1 - H_2\|_{L^\infty(\mathbb{R}^N)} \quad \text{by the previous argument.}
\end{aligned}
$$

Hence, letting $R \to \infty$, we have

$$
\|u - w\|_{L^\infty(\mathbb{R}^N)} \le \|H_1 - H_2\|_{L^\infty(\mathbb{R}^N)}.
$$

This completes the proof. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. In addition to the equation in the statement of Theorem 1.1, consider

$$w(x) + H_2(\nabla w(x)) = f(x), \quad x \in \mathbb{R}^N.$$

Then, by Lemma 3.1 and Lemma 2.3,

$$\begin{aligned} \|u - v\|_{L^\infty(\mathbb{R}^N)} &\le \|u - w\|_{L^\infty(\mathbb{R}^N)} + \|w - v\|_{L^\infty(\mathbb{R}^N)} \\ &\le \|H_1 - H_2\|_{L^\infty(\mathbb{R}^N)} + \|f - g\|_{L^\infty(\mathbb{R}^N)}. \end{aligned}$$

This completes the proof. □

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