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ON THE EXISTENCE OF EQUILIBRIUM PRICE

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1. Introduction

The Debreu-Gale-Nikaido theorem [2] is a potential tool to prove the existence of a market equilibrium price. Walras' law is of a quantitative nature (i.e. it measures the value of the total excess demand), and it is interesting to note that the existence result holds true under some qualitative assumptions. In fact, the Debreu-Gale-Nikaido theorem states that the continuity of the excess demand function and Walras' law has the following implication : For some price and corresponding value of the excess demand function, it is not possible to respond with a new price system such that the value at the new price of every element in the value of the demand function associated with the old price system is strictly positive.

Smale [5] gave the equilibrium price of a pure exchange economy and unified the existence, algorithm and dynamic questions of the economy by using the Sard implicit function theorem. In a recent paper, Tarafdar-Thompson [6] have given several proofs on the existence of the equilibrium price, which is equivalent to Smale's result, by using the variational inequality, degree theory and Brouwer's fixed point theorem.

The purpose of this note is twofold. First, we shall give a simple proof of the existence of the equilibrium price by using the classical Knaster- Kuratowski-Mazurkiewicz theorem. Next, we shall give a new kind of existence result of the market equilibrium price using Kim's intersection theorem [4].

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2. Preliminaries

Let A be a subset of a topological space X. We shall denote by 2^A the family of all subsets of A. If A is a non-empty subset of R^n, we shall denote by $co\,A$ the convex hull of A. Let $\Delta_1 = \{p = (p_1, \cdots, p_n) \in R^n \mid p_i \geq 0$ for each $i = 1, \cdots, n$ and $\Sigma_{i=1}^n p_i = 1\}$ be the standard price simplex and $\Delta_0 = \{p = (p_1, \cdots, p_n) \in R^n \mid \Sigma_{i=1}^n p_i = 0\}$. Let $e_i = (0, \cdots, 0, \underbrace{1}_{i-th}, 0, \cdots, 0)$ denote the i-th unit price vector in Δ_1 for each $i = 1, \cdots, n$ and O denote the zero price vector without any confusion ; then for any price vector $p \in \Delta_1$, there exist $\lambda_1, \cdots, \lambda_n \in R^1$ such that

$$p = \sum_{i=1}^n \lambda_i e_i \quad \text{where} \quad \sum_{i=1}^n \lambda_i = 1,\ 0 \leq \lambda_i \leq 1 \text{ for each } \ i = 1, \cdots, n.$$

The model of the exchange economy under consideration is as follows : Let n be the number of commodities; a price system $p = (p_1, \cdots, p_n)$, $p_i \geq 0$, where p_i represents the price of the unit of the i-th commodity; two functions $D, S : R^n_+ \setminus \{0\} \to R^n_+$ is called the demand and the supply function respectively, where

$$R^n_+ = \{x = (x_1, \cdots, x_n) \in R^n \mid x_i \geq 0 \text{ for each } i = 1, \cdots, n\}.$$

The excess demand function $\xi : R^n_+ \setminus \{0\} \to R^n$ is defined by

$$\xi(p) := D(p) - S(p) \quad \text{for each } p \in R^n_+ \setminus \{0\}.$$

The price p at which $\xi(p) = O$ is called the *market equilibrium price.* We shall use the notation $\xi(p) = (\xi_1(p), \cdots, \xi_n(p))$, where $\xi_i(p)$ is the excess demand for the i-th commodity at the price p.

The following theorem is essential in proving our main result:

LEMMA 1. (Knaster-Kuratowski-Mazurkiewicz [1]) *Let X be the set of vertices of a simplex in R^n, and let $F : X \to 2^E$ be a compact valued multimap such that $co\{x_1, \cdots, x_n\} \subset \cup_{i=1}^n F(x_i)$ for each finite subset $\{x_1, \cdots, x_n\} \subset X$. Then $\cap_{x\in X} F(x) \neq \emptyset$.*

Next, the following result is the open set version of Lemma 1 :

LEMMA 2. (Kim [4]) *Let X be a non-empty subset of a topological vector space E, and let $G : X \to 2^E$ be an open set-valued multimap satisfying the KKM condition, i.e.*

$$co\{x_1, \cdots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i) \quad \text{for each finite subset} \quad \{x_1, \cdots, x_n\} \subset X.$$

Then the family $\{G(x) \mid x \in X\}$ of open sets has the finite intersection property.

3. Existence of equilibrium price

We begin with the following result due to Tarafdar-Thompson [6], which is equivalent to Smale's result [5]. They proved it by using the (Brouwer) degree theory and the Brouwer fixed point theorem separately; but we shall give a simple proof of the following result by using Lemma 1.

THEOREM 1. *Let $\xi : \Delta_1 \to \Delta_0$ be a continuous excess demand function such that for any price $p = (p_1, \cdots, p_n) \in \Delta_1$ with $p_k = 0$, $\xi_k(p) \geq 0$.*

Then there exists a market equilibrium price $\hat{p} \in \Delta_1$ such that $\xi(\hat{p}) = O$.

Proof. For each $i \in I = \{1, \cdots, n\}$, we first define a subset F_i of Δ_1 by

$$F_i := \{p \in \Delta_1 \mid \xi_i(p) \leq 0\}$$

Since ξ_i is continuous and $\xi(p) \in \Delta_0$ for all $p \in \Delta_1$, by the assumption, F_i is a non-empty closed (compact) subset of Δ_1 for each $i \in I$. Furthermore, by the assumption again, the collection $\{F_i \mid i \in I\}$ of closed sets satisfies the KKM condition, i.e., for every non-empty subset J of I, $co\{e_j \mid j \in J\} \subset \cup_{j \in J} F_j$. In fact, for any price $p \in co\{e_j \mid j \in J\}$, there exist $0 \leq \lambda_j \leq 1$ for each $j \in J$ such that $p = \Sigma_{j \in J} \lambda_j e_j$.

Since $\xi_i(p) \geq 0$ for each $i \notin J$, $\xi_j(p) \leq 0$ for each $j \in J$; and hence $co\{e_j \mid j \in J\} \subset \cup_{j \in J} F_j$.

Therefore, by Lemma 1, we have $\cap_{i=1}^n F_i \neq \emptyset$; so that there exists $\hat{p} \in \cap_{i=1}^n F_i \subset \Delta_1$. Since $\Sigma_{i=1}^n \xi_i(\hat{p}) = 0$ and $\xi_i(\hat{p}) \leq 0$ for each $i = 1, \cdots, n$, we have $\xi(\hat{p}) = (\xi_1(\hat{p}), \cdots, \xi_n(\hat{p})) = O$. This completes the proof.

EXAMPLE. Let $\xi(p_1, p_2) := (\frac{3}{2} - 2p_1 - p_2, p_1 - \frac{1}{2})$ be a continuous excess demand function on Δ_1. Then ξ satisfies the assumption of Theorem 1, so that there exists a market equilibrium price $\hat{p} = (\frac{1}{2}, \frac{1}{2}) \in \Delta_1$ such that $\xi(\hat{p}) = O$.

Now we give a new existence theorem of market equilibrium price by using Lemma 2.

THEOREM 2. *Let $\xi : \Delta_1 \to R^n$ be a continuous excess demand function. Suppose that there exists some $\varepsilon > 0$ satisfying the following conditions :*

(1) for each $i \in \{1, \cdots, n\}$, there exists $p \in \Delta_1$ such that $\xi_i(p) < \varepsilon$;

(2) for any price $p = (p_1, \cdots, p_n) \in \Delta_1$ with $p_k = 0$, there exists $0 < \varepsilon' < \varepsilon$ such that $\xi_k(p) > \varepsilon'$;

(3) If $\frac{1}{n}\sum_{i=1}^{n} \xi_i(p) < \varepsilon$, then $\xi_i(p) = 0$ for each $i = 1, \cdots, n$.

Then there exists a market equilibrium price $\hat{p} \in \Delta_1$ such that $\xi(\hat{p}) = O$.

Proof. For each $i \in I = \{1, \cdots, n\}$, we define a subset G_i of Δ_1 by

$$G_i := \{p \in \Delta_1 \mid \xi_i(p) < \varepsilon\}$$

Since ξ_i is continuous and the assumption (1), each G_i is a non-empty open subset of Δ_1 for each $i \in I$. We now show that the collection $\{G_i \mid i \in I\}$ of open sets satisfies the KKM condition, i.e., for every non-empty subset J of I, $co\{e_j \mid j \in J\} \subset \cup_{j\in J} G_j$. In fact, for any price $p \in co\{e_j \mid j \in J\}$, there exist $0 \leq \lambda_j \leq 1$ for each $j \in J$ such that $p = \Sigma_{j\in J}\lambda_j e_j$. By the assumption (2), $\xi_i(p) > \varepsilon'$ for each $i \notin J$, so that $\xi_j(p) \leq \varepsilon' < \varepsilon$ for each $j \in J$; and hence $co\{e_j \mid j \in J\} \subset \cup_{j\in J} G_j$.

Therefore, by Lemma 2, we have $\cap_{i=1}^{n} G_i \neq \emptyset$; so that there exists $\hat{p} \in \cap_{i=1}^{n} G_i \subset \Delta_1$, i.e., $\xi_i(\hat{p}) < \varepsilon$ for each $i = 1, \cdots, n$. Since $\Sigma_{i=1}^{n} \xi_i(\hat{p}) < n\varepsilon$, by the assumption (3), we have $\xi(\hat{p}) = (\xi_1(\hat{p}), \cdots, \xi_n(\hat{p})) = O$. This completes the proof.

REMARKS. (i) The assumption (2) is more natural in real market economy than the assumption in Theorem 1. In fact, when $p_k = 0$, there may be some positive excess demand $\xi_k(p) > 0$.
(ii) By the assumption (2), the market equilibrium price $\hat{p}$ must lie in the interior of the price simplex Δ_1.

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