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CANCELLATION OF LOCAL SPHERES WITH RESPECT TO WEDGE AND CARTESIAN PRODUCT

HANS SCHEERER AND HEE-JIN LEE

0. Introduction

Let $\mathcal{C}$ be a category of (pointed) spaces. For $X, Y \in \mathcal{C}$ we denote the wedge (or one point union) by $X \vee Y$ and the cartesian product by $X \times Y$. Let $Z \in \mathcal{C}$; we say that Z cancels with respect to wedge (resp. cartesian product) and $\mathcal{C}$, if for all $X, Y \in \mathcal{C}$ the existence of a homotopy equivalence $X \vee Z \to Y \vee Z$ implies the existence of a homotopy equivalence $X \to Y$ (resp. for cartesian product). If this does not hold, we say that there is a non-cancellation phenomenon involving Z (and $\mathcal{C}$).

Non-cancellation phenomena are studied in various papers. We refer to [5], [8] and [9] for further information in the case of the wedge and to [6], [13] for the case of the product. For $\mathcal{C}$ the category of 1-connected rational CW-spaces the question has been studied in [2], [3], [4]. But even in this case it seems not to be known whether cancellation always holds.

Let $R \subseteq \mathbb{Q}$ be a subring. A space X is called R-local, if its reduced homology $\tilde{H}_*(X; Z)$ is an R-module. Any simply connected CW-space X has an R-localization denoted by X_R (see [7]). Let $\mathcal{C}$ be the category of spaces of the homotopy type of CW-complexes; let $1-\mathcal{C}$ (resp. $1-\mathcal{C}_R$) be the subcategory of simply connected (resp. simply connected R-local) spaces.

In the present paper we shall in particular prove the following results in a rather elementary way:

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RESULT I. *Let $R = \mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ away from $\{p\}$, p a prime.*

(1) *Then S^n_R, $n \geq 2$, cancles with respect to the wedge and $1 - \mathcal{C}_R$.*

(2) *Then $K(R, n)$, $n \geq 2$, cancels with respect to the cartesian product and the subcategory of $1 - \mathcal{C}_R$ of spaces of finite type over R.*

RESULT II. *Denote by p the smallest prime not invertable in R. Then the local sphere S^n_R, $n \geq 2$, cancesl with respect to cartesian product and the subcategory of $1 - \mathcal{C}_R$ of $R - CW$-spaces X which have finitely generated homology $H_*(X; R)$ and R-dimension$(X) \leq n + 2p - 4$, provided*

(a) *either n is odd and $R = \mathbb{Z}_{(p)}$,*

(b) *or n is even.*

Partially these results were first found by applying algebraizations of tame homotopy theory (in the form of [10] in case of the cartesian product and in the form of [11] in case of the wedge). For the product such an approach is pursued in [1]. In fact, some of the technical calculations below are adapted from [1]. We also note that Result II is still in the spirit of tame homotopy theory.

In Section 1 we shall give a basic simple lemma. In Section 2 we shall more generally discuss wedge cancellation of Moore spaces (resp. cartesian product cancellation of Eilenberg-MacLane spaces) to obtain Result I as a special case. Result II will be proved in Section 3.

1. A basic lemma

Let $R = \mathbb{Z}_{(p)}$ (resp. $R = \mathbb{Z}/p\mathbb{Z}$), let A, B, U be R-modules and U monogenic, e.g. $U \cong R$ or $U \cong \mathbb{Z}/p^k\mathbb{Z}$ for some k, in case $R = \mathbb{Z}_{(p)}$.

Assume that an isomorphism $\varphi : A \oplus U \to B \oplus U$ is given. We then do not only want to know that A and B are isomorphic, we would like to construct an isomorphism $\hat{\varphi} : A \to B$ from φ in such a way that in the applications the construction can be realized geometrically. We paraphrase this as follows:

LEMMA. *There is a "good " way to construct an isomorphism $\hat{\varphi}$: $A \to B$ from φ.*

Proof. Let $u \in U$ be a generator. We write:

(i) $\varphi(\lambda u + a) = u,\ \lambda \in R,\ a \in A$;
(ii) $\varphi(u) = \tau u + b,\ \tau \in R,\ b \in B$.

Case 1. Let $\tau \in R^*$, the set of unit of R.

Define $\beta : B\oplus U \to B\oplus U$ by setting $(\beta|B)$ the inclusion $B \to B\oplus U$ and $\beta(u) = -\tau^{-1}b + \tau^{-1}u$. (Note that u and $-\tau^{-1}b + \tau^{-1}u$ have the same order). Then β is an isomorphism and $\beta \circ \varphi(u) = u$. Hence the composition $A \hookrightarrow A\oplus U \xrightarrow{\beta\circ\varphi} B\oplus U \xrightarrow{pr} B$(where pr is the projection) is an isomorphism, because $\beta\circ\varphi$ induces an isomorphism on the quotients $(A \oplus U)/U \to (B \oplus U)/U$.

Case 2. Let $\lambda \in R^*$.

In this case $\tilde\varphi := (pr \circ \varphi)|A$ is already an isomorphism. To see this define $\alpha : A \oplus U \to A \oplus U$ by setting $\alpha|A$ the inclusion $A \to A \oplus U$ and $\alpha(u) = \lambda u + a$. Then α is an isomorphism and $\varphi \circ \alpha(u) = u$. Then, as above, $A \hookrightarrow A \oplus U \xrightarrow{\varphi\circ\alpha} B \oplus U \xrightarrow{pr} B$ is an isomorphism, it coincides with $(pr \circ \varphi)|A$.

Case 3. Let $\lambda, \tau \notin R^*$ (e.g. $\lambda = \tau = 0$ in case $R = \mathbb{Z}/p\mathbb{Z}$).

Then we have $\mu := \tau(1-\lambda) + 1 \in R^*$ and $(1-\lambda) \in R^*$. Define $\beta : B \oplus U \to B$ by $\beta|B = id_B$ and setting $\beta(u) = -\mu^{-1}(1-\lambda)b$; set $\tilde\varphi := (\beta \circ \varphi)|A$.

We claim that $\tilde\varphi$ is an isomorphism.

Injectivity: Let $z \in A$ with $\tilde\varphi(z) = 0$. The kernel of β is generated by $b + \mu(1-\lambda)^{-1}u$. Hence $\varphi(z) = k(b + \mu(1-\lambda)^{-1}u)$ for some $k \in R$. On the other hand

$$\begin{aligned}\varphi(u+a) &= \tau u + b + u - \lambda(\tau u + b)\\ &= \mu u + (1-\lambda)b\end{aligned}$$

Hence, $\varphi(k(1-\lambda)^{-1}(u+a)) = \varphi(z)$. Since φ is injective, we have $k(1-\lambda)^{-1}(u+a) = z$; this implies $k = 0$, if the order of u is infinite, and k a multiple of order(u) in the other case; in both cases it follows $z = 0$ (because order (a) is not larger than order (u)).

Surjectivity: Let $w \in B$ and choose $k \in R, v \in A$ such that

$\varphi(ku + v) = w$. Then

$$\begin{aligned}\varphi(v - ka) &= w - \varphi(ku) - \varphi(ku)\\ &= w - \varphi(k(u + a))\\ &= w - k\mu u - k(1 - \lambda)b.\end{aligned}$$

It follows $\tilde{\varphi}(v-ka) = \beta\circ\varphi(v-ka) = w + k\mu\mu^{-1}(1-\lambda)b - k(1-\lambda)b = w$.

2. Proof of Result I

We first fix some more notation.

A map $f : X \to Y$ is called an "R-homology equivalence" (resp. "R-cohomology equivalence"), if $H_*(f;R)$ (resp. $H^*(f;R)$) is an isomorphism. For $X, Y \in 1 - \mathcal{C}_R$ (resp. $X, Y \in 1 - \mathcal{C}_R$ of finite type over R) f is then a homotopy equivalence by the Whitehead theorem.

Let V be an abelian group. A Moore space $M(V,n)$, $n \geq 2$, is a simply connected space with reduced homology $\tilde{H}_i(M(V,n);\mathbb{Z}) = 0$ for $i \neq n$ and $H_n(M(V,n);\mathbb{Z}) \cong V$.

THEOREM 1. *Let $R = \mathbb{Z}_{(p)}$, let V be a finitely generated R-module and set $M := M(V,n), n \geq 2$. Let $X, Y \in 1 - \mathcal{C}$ and suppose that an R-homology equivalence*

$$\phi : X \vee M \to Y \vee M$$

is given. Then there exists an R-homology equivalence $X \to Y$.

COROLLARY. *The spaces S^n_R and $M(\mathbb{Z}/p^k\mathbb{Z}, n), n \geq 2$, cancel with respect to the wedge and $1 - \mathcal{C}_R$.*

Proof of Theorem 1. Note that R is a principal ideal ring, hence V is a finite direct sum of monogenic R-modules. Without loss of generality we may therefore assume that V is monogenic, i.e. either $V \cong R$ or $V \cong \mathbb{Z}/p^k\mathbb{Z}$ for some k.

Recall that $\tilde{H}_i(X \vee M; R) \cong \tilde{H}_i(M;R) \oplus \tilde{H}_i(M;R)$ for all i. Therefore, for $i \neq n$, the isomorphisms $H_i(\phi;R)$ can be identified with the homomorphisms induced by the composition $X \to X \vee M \xrightarrow{\phi} Y \vee M \to Y$. We now want to derive $\tilde{\phi} : X \to Y$ from ϕ, such that $H_n(\tilde{\phi};R)$

is an isomorphism by realizing the constructions in the Basic Lemma geometrically and keeping $H_i(\tilde{\phi}; R) = H_i(\phi; R)$ for $i \neq n$.

For convenience, we set $A := H_n(X; R), B := H_n(Y; R), U := H_n(M; R)$ (note that U is canonically isomorphic to V) and $\varphi := H_n(\phi; R)$. Denote by u a generator of U and write

(i)$\varphi(\lambda u + a) = u, \lambda \in R, a \in A$,

(ii) $\varphi(u) = \tau u + b, \tau \in R, b \in B$.

For the sake of simplicity we discuss only Case 3.

For any pointed space Z, let $[M, Z]$ denote the group of pointed homotopy classes of pointed maps $M \to Z$. Define a homomorphism $h : [M, Z] \to H_n(Z, R)$ by $[\psi] \mapsto \psi_*(u)$ for $[\psi] \in [M, Z], u \in H_n(M, R)$ as above. Note that $[M, M]$ is a (nilpotent) R-local group, because M is an R-local suspension. Hence image (h) is an R-module for all Z.

Since $u \in H_n(X \vee M; R)$ is in the image of h, so is $\varphi(u) = \tau u + b \in H_n(Y \vee M; R)$ by naturality of h. Moreover, u and $\tau u \in H_n(Y \vee M; R)$ lie in the image of h. hence we have $b \in$ image (h). Choose $[\psi] \in [M, Y]$ such that $h([\psi]) = b$. Define $\bar{\beta} : Y \vee M \to Y$ by $\bar{\beta}|Y = id_Y$ and $\bar{\beta}|M := -\mu^{-1}(1 - \lambda)\psi$; set $\tilde{\phi} := \bar{\beta} \circ \phi|X$. We then have $\bar{\beta}_*(u) = -\mu^{-1}(1 - \lambda)b$ and the construction in the Basic Lemma implies that $H_n(\tilde{\phi}; R)$ is an isomorphism; in degrees $i \neq n$ we have $H_i(\tilde{\phi}; R) = H_i(\phi; R)$, hence $\tilde{\phi}$ is an R-homology equivalence.

THEOREM 2. *Let $R = \mathbb{Z}_{(p)}$, let V be a finitely generated R-module and let $X, Y \in 1 - \mathcal{C}$ be of finite type over R. Assume that an R-cohomology equivalence $X \times K(V, n) \to Y \times K(V, n)$ is given. Then there exists an R-cohomology equivalence $X \to Y$.*

As a corollary we obtain part (2) of Result I.

Proof. As above we may suppose that V is monogenic . Let $\phi : X \times K(V, n) \to Y \times K(V, n)$ be an R-cohomology equivalence. (Note that ϕ is then also an R-homology equivalence and that all $\pi_i(\phi) \otimes R$ are isomorphisms[7]). For $i \neq n$ we may identify the isomorphisms $\pi_i(\phi) \otimes R$ with the corresponding induced homomorphisms of the composition $X \to X \times K(V, n) \xrightarrow{\phi} Y \times K(V, n) \xrightarrow{pr} Y$. From ϕ we will now construct a map $\tilde{\phi} : X \to Y$ such that $\pi_i(\tilde{\phi}) \otimes R = \pi_i(\phi) \otimes R$ for $i \neq n$ and such that $H^i(\tilde{\phi}; \mathbb{Z}/p\mathbb{Z})$ is an isomorphism for $i \leq n$; then, in particular, $\pi_n(\tilde{\phi}) \otimes R$ is surjective (see [7]). From $\pi_n(X) \otimes R \cong$

$\pi_n(Y) \otimes R$ it then follows that $\pi_n(\tilde{\phi}) \otimes R$ is an isomorphism; hence $\tilde{\phi}$ is an R-homology equivalence.

Set $\varphi := H^n(\phi; \mathbb{Z}/p\mathbb{Z}), A := H^n(Y; \mathbb{Z}/p\mathbb{Z}), B := H^n(X; \mathbb{Z}/p\mathbb{Z}), U := H^n(K(V,n); \mathbb{Z}/p\mathbb{Z})$ with generator $u \in U$. Note that $H^n(X \times K(V,n); \mathbb{Z}/p\mathbb{Z}) \cong B \oplus U, H^n(Y \times K(V,n); \mathbb{Z}/p\mathbb{Z}) \cong A \oplus U$. Write

(i) $\varphi(\lambda u + a) = u, \lambda \in \mathbb{Z}/p\mathbb{Z}, a \in A$,

(ii) $\varphi(u) = \tau u + b, \tau \in \mathbb{Z}/p\mathbb{Z}, b \in B$.

Let us again only consider Case 3 ; note that the base ring is now $\mathbb{Z}/p\mathbb{Z}$, hence we have $\lambda = \tau = 0$. Define a map $\beta : X \to X \times K(V,n)$ such that the first component $\beta_1 = id_X$ and second component $\beta_2 : X \to K(V,n)$ satisfies $\beta_2^*(u) = -b$. Such a map β_2 exists by the following reasoning : Let $r : V \to \mathbb{Z}/p\mathbb{Z}$ be reduction mod p. Clearly, $u = r_*(\tilde{u})$ where $\tilde{u} \in H^n(K(V,n); V)$ is the fundamental class. Hence $\varphi(u) = b = r_*(H^n(\phi; V)(\tilde{u}))$. Observe that $H^n(X \times K(V,n); V)$ admits a canonical direct sum decomposition as $H^n(X; V) \oplus H^n(K(V,n); V)$. Writing correspondingly $H^n(\phi; V)(\tilde{u}) = \tilde{u}_1 + \tilde{u}_2$ we have $r_*(\tilde{u}_1) = b$ and $r_*(\tilde{u}_2) = 0$. Choose β_2 as the map $X \longrightarrow K(V,n)$ including $-\tilde{u}_1$. Then, according to the Basic Lemma $\tilde{\phi} := pr \circ \phi \circ \beta : X \to Y$ induces an isomorphism $H^n(\tilde{\phi}; \mathbb{Z}/p\mathbb{Z})$ whereas $H^i(\tilde{\phi}; \mathbb{Z}/p\mathbb{Z}) = H^i(\varphi; \mathbb{Z}/p\mathbb{Z})$ for $i < n$. Thus $\tilde{\phi}$ is as required above

3. Proof of Result II

We first recall shortly the notion of R-dimension. A 1-connected R-local CW-complex of R-dimension m is built from a point by successivly attaching reduced cones on R-local spheres $S_R^n, 1 \leq n < m$.

Let n be odd, $n \geq 3$ and let $R = \mathbb{Z}_{(p)}$. For any $X \in 1 - C_R$ denote by $P^k(X)$ the k-th Postnikov section of X. Note that $P^m(S_R^n)$ is an Eilenberg-MacLane space $K(R,n)$ for $m = n + 2p - 4$ by [12].

Let X, Y be 1-connected R-local CW-complexes of finite type over R and with R-dimension(X), R-dimension(Y) $\leq m$. Let $X \times S_R^n$ and $Y \times S_R^n$ be homotopy equivalent. Then $P^m(X \times S_R^n) \backsim P^m(X) \times P^m(S_R^n) \backsim P^m(Y \times s_R^n) \backsim P^m(Y) \times P^m(S_R^n)$. By Theorem 2 $P^m(X)$ and $P^m(Y)$ are homotopy equvalent. It then follows $X \backsim Y$ (by the condition on the dimensions).

Thus part (1) is proved.

Let now n be even, $n \geq 2$, and let a homotopy equivalence $\phi : X \times S^n_R \to Y \times S^n_R$ be given, $X, Y \in 1 - \mathcal{C}_R$.

Hilfssatz: Let $u \in H^n(S^n_R; R)$ be a generator. Write $H^n(\phi, R)$ $(\lambda u + b) = u$ with $\lambda \in R, b \in H^n(Y; R)$ and $H^n(\phi; R)(u) = \tau u + a, \tau \in R$, $a \in H^n(X, R)$. Then, if $\lambda \neq 0$ (resp. $\tau \neq 0$) we have $\lambda \in R^*$ (resp. $\tau \in R^*$) and $H^n(\phi; R)(u) = \sigma u$ with $\sigma \in R^*$.

Proof. Assume $\lambda \neq 0$ (the case $\tau \neq 0$ is similar). Then $0 = H^{2n}(\phi; R)(\lambda u + b)^2 = H^{2n}(\phi; R)(\lambda^2 u^2 + 2\lambda u b + b^2) = H^{2n}(\phi; R)(2\lambda u b + b^2)$. Hence $2\lambda u b + b^2 = 0$; this implies $2\lambda b = 0$ and $b^2 = 0$. As a consequence $H^n(\phi; R)(2\lambda^2 u) = 2\lambda u - H^n(\phi; R)(2\lambda b) = 2\lambda u$. We deduce $\lambda \in R^*$ and $H^n(\phi; R)(u) = \sigma u$ with $\sigma \in R^*$.

PROPOSITION. *Let in addition $H^*(X; R)$ and $H^*(Y; R)$ be finitely generated R-modules. Suppose that the situation of the Hilfssatz applies for a homotopy equivalence $\phi : X \times S^n_R \to Y \times S^n_R$. Then there is a homotopy equivalence $X \to Y$.*

Proof. We have $H^*(X \times S^n_R) \cong H^*(X; R) \oplus H^*(X; R) \cdot u$ and similarly for $H^*(Y \times S^n_R; R)$. We assume $H^*(\phi; R)(u) = \sigma u$. It follows that $H^*(\phi; R)$ maps $H^*(Y; R) \cdot u$ into $H^*(X; R) \cdot u$. As R-modules $H^*(X; R)$ and $H^*(X; R) \cdot u$ are isomorphic. Hence $H^*(\phi; R)$ induces an isomorphism of quotient modules $H^*(Y; R) \cong H^*(Y \times S^n_R)/H^*(Y; R) \cdot u$ and $H^*(X; R) \cong H^*(X \times S^n_R)/H^*(X; R) \cdot u$. Therefore the map $X \hookrightarrow X \times S^n_R \xrightarrow{\phi} Y \times S^n_R \xrightarrow{pr} Y$ is an R-cohomology equivalence.

It remains to study the case $\lambda = \tau = 0$, that is

(i) $H^n(\phi; R)(b) = u, b \in H^n(Y; R)$,

(ii) $H^n(\phi; R)(u) = a, a \in H^n(X; R)$.

Note that the Basic Lemma, Case 3, can be applied here without the assumption $R = \mathbb{Z}_{(p)}$, because $\mu = (1 - \lambda) = 1$ in this simple situation. But we still have to check geometric realizability. Here we need the assumption R-dimension(X), R-dimension$(Y) \leq m = n + 2\bar{p} - 4$.

Suppose first $m < 2n-1$; then $P^m(S^n_R) \backsim K(R, n)$ and the argument can be completed as in the proof of part (1).

Assume now $m \geq 2n-1$. Then $P^m(S^n_R)$ fits into the fibre square

$$\begin{array}{ccc} P^m(S^n_R) & \longrightarrow & E \\ \downarrow & & \downarrow \\ K(R,n) & \xrightarrow{\iota^2} & K(R,2n) \end{array}$$

where $E \to K(R,2n)$ is the path fibration (see [12]) and where ι^2 denotes the map inducing the square of the fundamental class in $H^n(K(R,n)); R)$.

Define a map $\beta = (\beta_1, \beta_2) : X \to X \times S^n_R$ by setting $\beta_1 = id_X$ and choosing β_2 such that $H^n(\beta_2; R)(u) = -a$. Such a map exists, because $a^2 = 0$ and R-dimension(X) is restricted. We claim that the composition $\psi : X \to Y$ of $X \xrightarrow{\beta} X \times S^n_R \xrightarrow{\phi} Y \times S^n_R \xrightarrow{pr} Y$ is an R-cohomology eqivalence. (In the following we will omit the coefficients R from the notation).

Injectivity of ψ^*:

Let $z \in H^l(Y)$ with $\psi^*(Z) = 0$. Set $\phi^*(z) = cu + c'$ with $c, c' \in H^*(X)$; then $\psi^*(z) = -ca + c' = 0$ and $c' = ca$.

Recall that $\phi^*(u+b) = a+u$.

Set $c = \phi^*(eu+e'), e, e' \in H^*(Y)$. Then $\phi^*(z) = c(a+u) = \phi^*((eu+e').(u+b)) = \phi^*(eub + e'u + e'b)$, i.e.

$$z = eub + e'u + e'b = u(be + e') + e'b.$$

It follows be $+e' = 0$, hence $z = e'b = -b^2e = 0$, because $b^2 = 0$.

Thus $H^l(\psi)$ is injective as long as $H^l(\phi)$ is.

Surjectivity of ψ^* :

Let $\omega \in H^l(X)$. Assume $\phi^*(xu + v) = \omega$ for $x, v \in H^*(Y)$, then

$$\begin{aligned} \phi^*(v - xb) &= \omega - \phi^*(xu) - \phi^*(xb) \\ &= \omega - \phi^*(x(u+b)) \\ &= \omega - (\phi^*(x))(a+u). \end{aligned}$$

Hence

$$\begin{aligned} \psi^*(v - xb) = \beta^*\phi^*(v - xb) &= \beta^*(\omega - (\phi^*(x))(a+u)) \\ &= \omega - (\beta^*\phi^*(x))(a-a) = \omega. \end{aligned}$$

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Hans Scheerer
Mathematisches Institut
Freie Universität Berlin
Arnimallee 3-5, 14195 Berlin, Germany

Hee-Jin Lee
Department of Mathematics Education
Chonbuk National University
Chonju, Chonbuk 561-756, Korea