J. Korean Math. Soc. **33** (1996), No. 1, pp. 1–14

STEEPEST DESCENT METHOD FOR LOCALLY ACCRETIVE MAPPINGS

C. E. Chidume

1. Introduction

Let E be a real normed linear space, $K \subseteq E$. A mapping $A : K \to E$ is called *strongly pseudocontractive* if there exists $t > 1$ such that the inequality

$$\|x - y\| \leq \|(1 + t)(x - y) - rt(Ax - Ay)\| \tag{1}$$

holds for all $x, y \in K$ and $r > 0$. If $t = 1$ then A is called *pseudocontractive.* The map A is called *locally* strongly pseudocontractive if each point of K has a neighbourhood N for which (1) holds for each $x, y \in N$ and some $t > 1$. Pseudocontractive operators have been studied by various authors (see e.g., [1], [2], [4], [8-12], [14], [16], [17], [18], [19], [21], [22], [28], [29], [30], [32-33], [37]). Interest in such mappings stems mainly from the fact that they are firmly connected with the important class of nonlinear *accretive operators.* A mapping U with domain $D(U)$ and range $R(U)$ in E is called *accretive* (see e.g., [2], [15]) if the inequality

$$\|x - y\| \leq \|x - y + t(Ux - Uy)\|$$

holds for each $x, y \in D(U)$ and all $t > 0$. medskip The accretive operators were introduced independently by Browder [3] and Kato [15]. If $E = H$, a Hilbert space, one of the earliest problems in the theory of accretive operators was to solve the equation $x + Ux = f$ for x, given an element f of H and an accretive operator U. We remark here that in Hilbert spaces, accretive operators are also called *monotone.* In [3],

Received April 18, 1994.

1991 AMS Subject Classification: 47H04, 47H05, 47H06 47H10, 47H15, 47H17.

Key words and phrases: Local strong pseudo-contractions; q-uniformly smooth spaces; Mann iteration process; Ishikawa iteration process.

Browder proved that if U is *locally* Lipschitzian and accretive then U is *m-accretive*, that is, $(I + U)$ is surjective. This result was subsequently generalized by Martin [20] to the continuous accretive operators.

The firm connection between the pseudocontractive mappings and the accretive operators is that a mapping U is pseudocontractive if and only if $(I - U)$ is accretive [3, Proposition 1] Consequently, the mapping theory for accretive operators is closely related to the fixed point theory of pseudocontractive operators.

It is well known (see for example, [4]) that many physically significant problems can be modelled in terms of an initial value problem of the form

$$(2) \qquad \begin{cases} \frac{dx}{dt} & = -Ux \\ x(0) & = x_0 \end{cases}$$

where U is either accretive or strongly accretive. Typical examples of how such evolution equations arise are found in models involving either the heat, the wave or the Schrödinger equation. Let $N(U)$ denote the kernel of U. We observe that members of $N(U)$ are, in fact, the equilibrium points of the system (2). Consequently, considerable effort has been devoted to developing constructive techniques for the determination of the kernels of accretive operators (see e.g., [5], [6], [7], [8-12], [13], [14], [22], [23-25], [27], [28], [29], [30], [32, 33], [35], [36], [37]). Moreover, since a continuous accretive operator can be approximated well by a sequence of strongly accretive ones, particular attention has been devoted to constructive techniques for the kernels of strongly accretive operators. In this connection, but in Hilbert space, Vainberg [35] and Zarantonello [39] introduced the steepest descent method:

$$(3) \qquad x_{n+1} = x_n - c_n U x_n, \qquad x_0 \in H, \quad n = 0, 1, 2, \ldots$$

and proved that if $U = I + T$ where T is a monotone Lipschitz map and $c_n = \lambda, n = 0, 1, 2, \ldots; \lambda$ a constant, then the sequence $\{x_n\}$ defined by (3) converges strongly to an element of $N(U)$. This result has been generalized and extended to more general Banach spaces (see e.g., [5], [8-12], [2], [23-26], [28], [29], [32], [33], [37]). Recently, the author proved the following theorem:

THEOREM 1 ([8]). *Suppose K is a nonempty closed bounded and convex subset of $L_p, p \geq 2$, and $T : K \to K$ is a Lipschitz strongly pseudocontractive mapping of K into itself. Let $\{c_n\}$ be a real sequence satisfying:*

(i) $0 < c_n < 1$ *for all* $n \geq 1$,
(ii) $\sum_{n=1}^{\infty} c_n = \infty$; *and*
(iii) $\sum_{n=0}^{\infty} c_n^2 < \infty$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by $x_1 \in K$,

$$x_{n+1} = x_n - c_n\, Ax_n, \quad n \geq 1 \tag{4}$$

converges strongly to a solution of the equation $Ax = 0$ where $A = I - T$.

Several authors have generalized and extended Theorem 1 in various directions. In [32], Schu extended the theorem to the class of continuous strongly pseudocontractive maps in real Banach spaces with property $(U, \alpha, m+1, m)$ (see e.g., [32] for definition). These Banach spaces include the L_p spaces, $p \geq 2$; and in [33] he extended the theorem to the class of *uniformly* continuous maps in *smooth* Banach spaces. Bethke [1] obtained a slight generalization of the theorem still in L_p spaces, $p \geq 2$; the author [10] and also Osilike [22] extended the theorem to the class of continuous strongly pseudocontractive maps on real uniformly smooth Banach spaces. Other generalizations can be found in Xu, Zhang and Roach [30]. The most general result for the global convergence of (4) for *strongly accretive maps* seems to be the main result of Xu and Roach [28] (see also a result of the author, [12]). A natural problem of interest (see e.g., [14], [37]) is to prove convergence theorems for approximating solutions of $Ax = 0$ when A is *locally* accretive and a solution is known to exist.

It is our purpose in this paper to prove that in real q-uniformly smooth Banach spaces (defined below) the steepest descent approximation method (4) converges strongly to a solution of the equation $Ax = 0$ (when one exists) for *locally* strongly accretive operators, A. In particular, our result (Theorem 2) will extend Theorem 1 to real q uniformly smooth Banach spaces (which include the L_p spaces, $1 < p < \infty$) and to the class of *locally* strongly pseudocontractive maps (see our Remarks 1 and 2). Furthermore, since Banach spaces with property

$(U, \alpha, m+1, m)$ are q-uniformly smooth, Theorem 2 also extends the result of Schu (Theorem 1 of [32]) to these more general Banach spaces and to operators which are continuous and *locally* strongly pseudocontractive, while Theorem 4 extends Theorem 2 of [32] to the class of *locally* Lipschitz continuous and strongly pseudocontractive maps. In addition, we shall prove a theorem (Theorem 3) on the convergence of the iteration process (4) to a solution of the equation $x + Ux = f$ where U is a continuous *locally* accretive map on a real q-uniformly smooth Banach space. This result is related to the results of Bruck [5], the author [9] and Carbone [6].

2. Preliminaries

Let E be a Banach space. We shall denote by J the normalised duality mapping from E to 2^{E^*} given by

$$Jx = \{f^* \in E^* \ : \ \|f^*\|^2 = \|x\|^2 = \langle x, f^* \rangle\}$$

where $\langle , \rangle$ denotes the generalized duality pairing. If E is uniformly convex then J is single-valued, and is uniformly continuous on bounded sets. In the sequel we shall denote single-valued normalized duality map by j.

Now, with $p > 1$, follwoing [38], we shall associate the generalized duality map J_p from E to E^* defined by

$$J_p(x) = \{f^* \in E^* \ : \ \langle x, f^* \rangle = \|x\|^p, \quad \text{and} \quad \|f^*\| = \|x\|^{p-1}\}.$$

In particular, J_2 is the usual normalized duality map on E. It is known (see e.g., [38]) that

$$(5) \qquad J_p(x) = \|x\|^{p-2} J(x) \quad \text{for} \quad x \neq 0.$$

Let E be a Banach space with $\dim E \geq 2$. The *modulus of smoothness* $\rho_E(\tau), \tau > 0$, of E is defined by

$$\rho_E(\tau) = \sup\{(\|x+y\| + \|x-y\|)/2 - 1 : x, y \in E, \ \ \|x\| = 1, \|y\| = \tau\}.$$

The Banach space E is *uniformly smooth* (see e.g., [34]) if $\lim_{\tau\to 0}\rho_E(\tau)/\tau = 0$, and E is called *q-uniformly smooth* (see e.g., [38]) if there exists a cosntant $c > 0$ such that

$$\rho_E(\tau) \le c\,\tau^q, \qquad 0 < \tau < \infty.$$

It is known (see e.g., [38], [34]) that

$$L_p \ \text{ is } \begin{cases} p-\text{uniformly smooth if } \ 1 < p \le 2 \\ 2-\text{uniformly smooth if } \ p \ge 2. \end{cases}$$

A Banach space E is called *smooth* (see e.g., [34], p.60) if, for every $x \in E$ with $\|x\| = 1$, there exists a unique $f^* \in E^*$ such that $\|f^*\| = f^*(x) = 1$. In [38], the following result which will be needed in the sequel is proved.

LEMMA 1 ([38]). *Let $q > 1$ be a real number and E be a smooth Banach space. Then the follwoing are equivalent:*

(i) *E is q-uniformly smooth;*

(ii) *There is a constant $c > 0$ such that for every $x, y \in E$, the following inequality holds;*

$$\|x + y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q \tag{6}$$

A mapping U is called locally strongly accretive if each point in the domain of U has a neighbourhood N for which there exist a constant $k > 0$ and $j(x-y) \in J(x-y)$ such that

$$\langle Ux - Uy,\ j(x-y)\rangle \ge k\|x-y\|^2. \tag{7}$$

holds for $x, y \in N$.

The following lemma has been proved:

LEMMA 2 ([37]). *Let E be a real Banach space, K a subset of E and $U : K \to E$. Then U is locally strongly pseudocontractive if and only if $(I - U)$ is a locally strongly accretive.*

3. Main results

In the sequel, c will denote the constant appearing in inequality (6). We prove the following theorems.

THEOREM 2. *Let E be a real q-uniformly smooth Banach space. Suppose T is a continuous locally strongly accretive map with open domain $D(T)$ in E and that $Tx = 0$ has a solution x^* in $D(T)$. Then there exist a neighbourbood B in $D(T)$ of x^* and a real number $r_1 > 0$ such that for any $r > r_1$ and some real sequence $\{c_n\}$, any initial guess $x_1 \in B$, the sequence $\{x_n\}$ generated from x_1 by*

$$x_{n+1} = x_n - c_n\, T\, x_n, \qquad n \geq 1, \tag{8}$$

remains in $D(T)$ and converges strongly to x^* with

$$\|x_n - x^*\| = O(n^{-(q-1)/q}).$$

Proof. Since T is locally strongly accretive, there exists a neighbourhood U of x^* such that for each $x \in U$,

$$\langle Tx - Tx^*,\ j(x - x^*)\rangle \geq k\|x - x^*\|^2.$$

Accretiveness of T on U implies T is locally bounded at each interior point of U (see e.g., Rockafellar [31], Reich [26]). So, we can choose $B = B_d(x^*)$, the closed ball of radius $d > 0, B \subseteq U$ so that $T(B)$ is bounded ***and*** T is strongly accretive on B. Let D be a constant such that $2d + \operatorname{diam}(T(B)) \leq D$. Let $r_1 = [c^{1/q}D]^{q/(q-1)}(dk)^{-q/(q-1)}$. Then $r_1 > 0$ and for $r \geq r_1$,

$$D \leq r^{(q-1)/q} dk\ c^{-q^{-1}}. \tag{9}$$

Let $c_n = \frac{1}{k(n+r)}, d_n = \frac{1}{k(n+r-1)^{(q-1)/q}}$.
Observe that $(1 - k\ c_n)^q d_n^q + c_n^q = d_{n+1}^q$.
Starting with an initial guess $x_1 \in B$, define the sequence $\{x_n\}_{n=1}^{\infty}$ inductively by (8).

Claim For all $n \geq 1, x_n$ is well defined and

$$\|x_n - x^*\| \leq d_n\ d\ r^{(q-1)/q}k.$$

The proof of this claim is by induction. For $n = 1, x_n$ is clearly in B. Suppose now that the claim has been proved for a particular choice of n. Then,

$$\|x_n - x^*\| \leq d_1\ d\ r^{(q-1)/q} k = d, \quad \text{so} \quad x_n \in B.$$

Thus, x_n is well defined by (8). Using (5), (6), (7) and the induction hypothesis, we obtain:

$$\|x_{n+1} - x^*\|^q = \|(1 - c_n)(x_n - x^*) + c_n(Sx_n - Sx^*)\|^q, \tag{10}$$

where $Sx := x - Tx$ for each $x \in B$. Observe that x^* is a solution of $Tx = 0$ and only if it is a fixed point of S. Moreover,

$$\begin{aligned}\langle Sx_n - Sx^*,\ J_q(x_n - x^*)\rangle &= \langle x_n - x^* - (Tx_n - Tx^*), J_q(x_n - x^*)\rangle \\ &= \|x_n - x^*\|^q - \langle Tx_n - Tx^*, J_q(x_n - x^*)\rangle \\ &\leq (1 - k)\|x_n - x^*\|^q.\end{aligned}$$

Hence, from (10), using (6):

(11)

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\leq (1 - c_n)^q\|x_n - x^*\|^q \\ &+ q\ c_n(1 - c_n)^{q-1}\langle Sx_n - Sx^*, J_q(x_n - x^*)\rangle + c\ c_n^q\|Sx_n - Sx^*\|^q \\ &\leq \left[(1 - c_n)^q + q(1 - k)c_n(1 - c_n)^{q-1}\right]\|x_n - x^*\|^q \\ &+ c\ c_n^q\|Sx_n - Sx^*\|^q,\end{aligned}$$

For $x \in (0, 1)$, consider the function

$$f(x) = (1 + x)^q, \quad q > 1$$

Then, there exists $\xi \in (0, x)$ such that

$$f(x) = f(0) + xf'(0) + x^2\frac{f''(\xi)}{2} = 1 + xq + \frac{x^2}{2}f''(\xi). \tag{i}$$

Observe that $f''(\xi) \geq 0$. Set $x = (1 - k)c_n(1 - c_n)^{-1}$ in (i) to get,

$$\left[1 + \frac{(1 - k)c_n}{1 - c_n}\right]^q = 1 + \frac{q(1 - k)c_n}{(1 - c_n)} + \frac{(1 - k)^2c_n^2}{(1 - c_n)^2}\frac{f''(\xi)}{2}$$

which simplifies to

$$[1-c_n+(1-k)c_n]^q$$
$$=(1-c_n)^q+q(1-k)c_n(1-c_n)^{q-1}+\frac{1}{2}(1-k)^2c_n^2(1-c_n)^{q-2}f''(\xi)$$

and implies (since $f''(\xi)\geq 0$):

$$(1-c_n)^q+q(1-k)c_n(1-c_n)^{q-1}\leq[1-c_n+(1-k)c_n]^q=(1-kc_n)^q$$

Hence, using this inequality, (11) yields:

$$\|x_{n+1}-x^*\|^q\leq(1-kc_n)^q\|x_n-x^*\|^q+c\,c_n^q\|Sx_n-Sx^*\|^q.$$

Observe that $\|Sx_n-Sx^*\|\leq D$ so that

$$\|x_{n+1}-x^*\|^q\leq(1-kc_n)^q\|x_n-x^*\|^q+c\,c_n^qD^q$$

which implies, by induction hypothesis

$$\|x_{n+1}-x^*\|^q\leq[(1-kc_n)^qd_n^q+c_n^q]d^q\ r^{q-1}\ k^q=d_{n+1}^qr^{q-1}\ k^q\ d^q$$

so that

$$\|x_{n+1}-x^*\|\leq d_{n+1}\ dk\ r^{(q-1)/q},$$

completing the induction process. Since $d_n=O(n^{-(q-1)/q})$, the error estimate of the theorem has also been established. This completes the proof.

COROLLARY 1. *Let E be a real q-uniformly smooth Banach space. Suppose U is a continuous locally strongly pseudocontractive map with open domain $D(U)$ in E and that U has a fixed point in $D(U)$. Then there exist a neighbourhood B in $D(U)$ of x^* and a real number $r_1>0$ such that for any $r>r_1$ and some real sequence $\{c_n\}$, any initial guess $x_1\in B$, the sequence $\{x_n\}$ generated from x_1 by*

$$x_{n+1}=x_n-c_n(I-U)x_n\qquad n\geq 1,$$

remains in $D(U)$ and converges strongly to x^* with

$$\|x_n-x^*\|=O(n^{-(q-1)/q}).$$

Proof. Follows immediately from Lemma 2 and Theorem 1.

REMARK 1. In [14], the author claimed to have generalized Theorem 1 to *locally* Lipschitzian and strongly pseudocontractive operators *in L_p spaces, $p \geq 2$*. He stated that if the mapping $U : D(U) \to E (E = L_p, p \geq 2)$ is locally Lipschitzian and strongly pseudocontractive, then there exists a closed region $B(x^*)$ containing a solution x^* of the equation $Tx = y$ such that, for arbitrary $x_0 \in B(x^*)$, the process $x_{n+1} = x_n + \lambda(y - Tx_n)$ for a suitable λ converges strongly to the solution x^*. However, as has already rightly been observed (MR. 92h:47090) the author fails to prove the existence of the region $B(x^*)$ where the iteration process is well defined. Moreover, there are several other inconsistencies in this result (see e.g., MR. 92h:47090).

REMARK 2. In [37], the author claimed to have extended Theorem 1 to general uniformly smooth Banach spaces E and to the class of *local* strongly pseudocontractive operators. He published the following theorem:

THEOREM XW ([37]). *Let K be a subset of a uniformly smooth Banach space E and $U : K \to E$ be a local pseudocontractive mapping. If $F(U) = \{x \in K : Ux = x\} \neq \emptyset$ and the range of U is bounded, then $\{x_n\} \subseteq K$ generated by $x_1 \in K$,*

$$x_{n+1} = x_n - c_n(I - U)x_n$$

with $\{c_n\} \subseteq (0,1]$, satsifying: $\sum_{n=1}^{\infty} c_n = \infty, c_n \to 0$, converges strongly to $x^* \in F(U)$ and $F(U)$ is a singleton set.

We remark immediately that the sequence $\{x_n\}$ in Theorem XW is not even well defined, as can be seen from the following easy example.

COUNTER-EXAMPLE TO THEOREM XW. *Take $E = \ell_2, K = \{x \in \ell_2 : \|x\| \leq 1\}$. Define $U : K \to E$ by*

$$U(x_1, x_2, x_3, \ldots) = (-4x_1, -4x_2, -4x_3, \ldots)$$

for arbitrary $(x_1, x_2, x_3, \ldots) \in K$ Then,

(i) *E is clearly uniformly smooth;*
(ii) *$Ux = x$ if and only if $x = 0$. Hence $F(U) \neq \emptyset$.*
(iii) *$\|Ux\| \leq 4$ for each $x \in K$. Hence, the range of U is bounded*
(iv) *$\langle (I - U)x - (I - U)y, j(x - y)\rangle = 5\|x - y\|^2$ for each $x, y \in K$.*

Now, choose $c_n = \frac{1}{n+1}, n = 1, 2, \cdots$ *and* $x_1 = (1, 0, 0, \ldots) \in K$. *Then* $x_2 = (-\frac{3}{2}, 0, 0, \ldots) \notin K$, *and so* x_3 *is not defined. In fact, the above choice of* x_1 *is not crucial. For example, for any* $\lambda \in (\frac{2}{3}, 1), x_1 = (\lambda, 0, 0, \ldots) \in K$ *and* $x_2 = (-\frac{3}{2}\lambda, 0, 0, \ldots) \notin K$. *Again* x_3 *is not defined. Other choices are obviously possible. This completes the counter-example.*

We now prove the follwoing theorem on the convergence of the steepest descent method to a solution of the equation $x + Tx = f$ for a locally accretive operator T in q-uniformly smooth Banach spaces.

THEOREM 3. *Let* E *be a real* q*-uniformly smooth Banach space. Suppose* T *is a continuous locally accretive map with open domain* $D(T)$ *in* E *and that* $f \in R(I+T)$. *Suppose the equation* $x + Tx = f$ *has a solution* $x^* \in D(T)$. *Then there exist a neighbourhood* $B \subseteq D(T)$ *of* x^* *and a real number* $r_1 > 0$ *such that for any* $r > r_1$, *any initial guess* $x_1 \in B$, *the sequence* $\{x_n\}_{n=1}^{\infty}$ *generated from* x_1 *by*

$$x_{n+1} = x_n - c_n(I - f + T)x_n, \qquad n = 1, 2, \ldots, \tag{12}$$

for some real sequence $\{c_n\}_{n=1}^{\infty}$ *remains in* $D(T)$ *and converges strongly to* x^* *with*

$$\|x_n - x^*\| = O(n^{-(q-1)/q}).$$

Proof. Let x^* denote a solution of the equation $x + Tx = f$. So, as in the proof of Theorem 2, we can choose $B = B_d(x^*)$, the closed unit ball of radius $d > 0, B \subseteq D(T)$ so that $T(B)$ is bounded and T is accretive on B. Let

$$r_1 = \left[C^{1/q} diam\ T(B)\right]^{q/(q-1)} d^{-q/(q-1)}.$$

Then $r > 0$ and diam $T(B) \leq r^{(q-1)/q} d\ c^{-q}$ for $r \geq r_1$. Let $c_n = \frac{1}{n+r}, d_n = \frac{1}{(n+r-1)^{(q-1)/q}}$ so that $(1 - c_n)^q d_n^q + c_n^q = d_{n+1}^q$. Starting with an initial guess $x_1 \in B$, define the sequence $\{x\}_{n=1}^{\infty}$ inductively by (12). As in the proof of Theorem 2, $\{x_n\}$ is well defined by (12). We now prove

$$\|x_n - x^*\| \leq d_n d\ r^{(q-1)/q}.$$

Now, using an induction argument as in the proof of Theorem 2, we have,

$$\|x_{n+1} - x^*\|^q \leq (1 - c_n)^q \|x_n - x^*\|^q - q\, c_n (1 - c_n)^{q-1} \langle Tx_n - Tx^*, J_q(x_n - x^*)\rangle + c\, c_n^q \|Tx^* - Tx_n\|^q.$$

Since $c_n(1 - c_n) \geq 0$ and T is accretive, it follows that

$$\|x_{n+1} - x^*\|^q \leq (1 - c_n)^q \|x_n - x^*\|^q + c\, c_n^q \|Tx^* - Tx_n\|^q.$$

Using the induction hypothesis and the fact that Tx_n and Tx^* belong to $T(B)$, the last inequality yields:

$$\|x_{n+1} - x^*\|^q \leq [(1 - c_n)^q d_n^q + c_n^q] d^q\, r^{(q-1)} = d_{n+1}^q\, d^q\, r^{(q-1)}$$

so that $\|x_{n+1} - x^*\| \leq d_{n+1} d\, r^{(q-1)/q}$, completing the induction argument and completing the proof of the theorem.

COROLLARY 2. *Let E be a real q-uniformly smooth Banach space. Suppose U is continuous locally pseudocontractive map with open domain $D(U)$ in E and that U has a fixed point x^* in $D(U)$. Then there exist a neighbourhood B in $D(U)$ of x^* and a real number $r_1 > 0$ such that for any $r > r_1$ and for some real sequence $\{c_n\}_{n=1}^{\infty}$, any initial guess $x_1 \in B$, the sequence $\{x_n\}_{n=1}^{\infty}$ generated from x_1 by*

$$x_{n+1} = x_n - c_n(I - U)x_n, \qquad n \geq 1,$$

remains in $D(U)$ and converges strongly to x^* with

$$\|x_n - x^*\| = O(n^{-(q-1)/q}).$$

Proof. Obvious, from Lemma 2 and Theorem 3.

ACKNOWLEDGMENTS. The author is grateful to Professor Abdus Salam, the International Atomic Energy Agency, UNESCO and the International Centre for Theoretical Physics, Trieste, for support.

References

1. M. Bethke, *Approximation von Fixpunkten streng pseudo-kontraktiver operatoren (Approxiamtion of fixed points of strongly pseudo-contractive operators)*, Math. Naturwiss Fak. **27** (1989), 263-270.
2. J. Bogin, *On strict pseudocontractions and a fixed point theorem*, Technion Preprint Series No.MT-219, Haifa, Israel, 1974.
3. F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc.(N. S.) **73** (1967), 875-882.
4. F. E. Browder, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, vol. 18, in Proc. Symp. Pure Math.; Part 2, Amer. Math. Soc., Providence, RI.
5. R. E. Bruck, *The iterative solution of the equation $y \in x + Tx$ for a monotone operator T in Hilbert space*, Bull. Amer. Math. Soc. (N. S.) **79** (1973), 1258-1262.
6. A. Carbone, *Iterative construction of solutions of functional equations involving multivalued operators in L^p spaces*, Jnanabha **20** (1990).
7. M. G. Crandall and A Pazy, *On the range of accretive operators*, Israel J. Math. **27** (1977), 235-246.
8. C. E. Chidume, *Iterative approximation of fixed points of Lipschitzian strictly pseudocontractive mappings*, Proc. Amer. Math. Soc. **99** (1987), 283-288.
9. C. E. Chidume, *The iterative solution of the equation $f \in x+Tx$ for a monotone operator T in L^p spaces*, J. Math. Anal. Appl **116** (1986), 531-537.
10. C. E. Chidume, *Convergence theorems for certain classes of nonlinear mappings*, ICTP, Trieste, Preprint IC/92/21 (1992).
11. C. E. Chidume, *Approximation of fixed points of strongly pseudo-contractive mapings*, Proc. Amer. Math. Soc. **120** (1994), 545-551.
12. C. E. Chidume, *Steepest descent approximations for accretive operator equations*, Nonlinear Anal. TMA **26** (1996), 299-311.
13. K. Deimling, *Zeros of accretive operators*, Manuscripta Math. **13** (1974), 283-288.
14. Zou, Bao Kang, *Iterative approximation of the solution of a locally Lipschitzian equation*, Appl. Math. Mech. (English Ed.) **12** (1991), 409-414; translated from Appl. Math. Mech. **12** (1991), 385-389 (Chinese). MR. 92h:47090.
15. T. Kato, *Nonlinear semigroups and evolution equations*, J. Math. Soc. Japan **19** (1967), 508-520.
16. W. A. Kirk, *A fixed point theorem for local pseudo-contract in uniformly convex spaces*, Manuscripta Math. **30** (1979), 89-102.
17. W. A. Kirk, *Remarks on pseudo-contractive mappings*, Proc. Amer. Math. Soc. **25** (1970), 820-823.
18. J. A. Gotica and W. A. Kirk, *Fixed point theorems for Lipschitzian pseudo-contractive mappings*, Proc. Amer. Math. Soc. **36** (1972), 111-115.
19. W. A. Kirk and C. Morales, *On the approximation of fixed points of locally nonexpansive mappings*, Canad. Math. Bull. **24** (1981), 441-445.

20. R. H. Martin, *A global existence theorem for autonomous differential equations in Banach spaces*, Proc. Amer. Math. Soc. **26** (1970), 307-314.
21. C. Morales, *Surjectivity theorems for multivalued mappings of accretive type*, Comment. Math. Univ. Carolin **26** (1985), 2.
22. M. O. Osilike, *Ishikawa and Mann iteration methods for nonlinear strongly accretive mappings*, Bull. Austral. Math. Soc. **46** (1992), 411-422.
23. S. Reich, *An iterative procedure for constructing zeros of accretive sets in Banach spaces*, Nonlinear Anal. **2** (1978), 85-92.
24. S. Reich, *Constructing zeros of accretive operators,II*, Appl. Anal. **9** (1979), 159-163.
25. S. Reich, *Constructive techniques for accretive and monotone operators*, in Applied Nonlinear Analysis (V. Lakshmikantham, Ed.), pp. 335-345; (Academic Press, New York, 1979).
26. S. Reich, *Strong convergence theorems of resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl **85** (1980), 287-292.
27. G. Nevanlinna and S. Reich, *Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces*, Israel J. Math. **32** (1979), 44-58.
28. Z. Xu and G. F. Roach, *A necessary and sufficient condition for convergence of steepest descent approximation to accretive operator equations*, J. Math. Anal. Appl **167** (1992), 340-354.
29. Z. Xu and G. F. Roach, *On the steepest descent approximation to solutions of nonlinear strongly accretive operator equations*, J. Comput. Math. (1992).
30. Z. Xu, Bo Zhang and G. F. Rozch, *On the steepest descent approximation to solutions of nonlinear strongly accretive operator equations*, J. Comput. Math. **7** (1992).
31. R. T. Rockafellar, *Local boundedness of nonlinear, monotone operators*, Michigan Math. J. **16** (1969), 397-407. MR. 40#6229.
32. J. Schu, *On a theorem of C. E. Chidume concerning the iterative approximation of fixed points*, Math. Nachr. **153** (1991), 313-319.
33. J. Schu, *Iterative construction of fixed points of strictly pseudocontractive mappings*, Applicable Analysis **40** (1991), 67-72.
34. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces, II-function spaces*, (Springer-Verlag, Berlin, 1979).
35. M. M. Vainberg, *On the convergence of the method of steepest descent for nonlinear equations*, Sibirsk. Math. Z. **2** (1961), 201-220.
36. M. M. Vainberg, *Variational methods and method of monotoneoperators in the theory of nonlinear equations*, (John Wiley and Sons, 1973).
37. Xinlong Weng, *Fixed point iteration for local strictly pseudo-contrtactive mappings*, Proc. Amer. Math. Soc. **113** (1991), 727-731.
38. Hong-Kun Xu. *Inequalities in Banach spaces withapplications*, Nonlinear Anal **16** (1991), 1127-1138.
39. E. H. Zarantonello, *The closure of the numerical rangecontains the specturm*, Bull. Amer. Math. Soc. (N.S.) **70** (1964), 781-783.

International Centre for Theoretical Physics
Trieste, Italy