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CONVERGENCE OF WAVELET EXPANSIONS AT DISCONTINUITY

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1. Introduction

Let ϕ be an orthogonal scaling function, i.e., $\phi : R \to R$ is a square integrable function having the properties:
1. the functions $\phi(x-n), n \in Z$, form an orthonormal system in $L^2(R)$.
2. the multiresolution subspaces V_m, of $L^2(R), m \in Z$ defined as the closed linear spans of orthogonal systems $\phi_{mn}(x) = 2^{\frac{m}{2}}\phi(2^m x - n), n \in Z$, are nested;

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$$

3. the union of the spaces $V_m, m \in Z$, is dense in $L^2(R)$.

Then by the first property the orthogonal projection P_m of $f \in L^2(R)$ onto V_m is given by

$$P_m f = \sum_{n \in Z} \langle f, \phi_{mn} \rangle \phi_{mn}, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denote the scalar product in $L^2(R)$. Because of the second and third property, the sequence $P_m f$ converges to f in the $L^2(R)$ norm as $m \to \infty$ for every $f \in L^2(R)$. In fact, $P_m f$ is a partial sum of the wavelet expansion associated with the given scaling function. Let ψ be a corresponding (mother)wavelet, i.e., a function in V_1 such that the system $\psi(x-n), n \in Z$, forms an orthonormal basis of the orthogonal complement of V_0 within V_1. Then the system $\psi_{mn}(x) =$

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$2^{\frac{m}{2}}\psi(2^m x-n), m,n \in Z$, is an orthormal basis of $L^2(R)$; every function $f \in L^2(R)$ admits the $L^2(R)$ convergent wavelet expansion

$$f = \sum_{m,n\in Z} \langle f, \psi_{mn}\rangle \psi_{mn}. \tag{1.2}$$

Now $P_m f$ is the partial sum

$$P_m f = \sum_n \sum_{k<m} \langle f, \psi_{kn}\rangle \psi_{kn}.$$

We assume, for some constant K,

$$|\phi(x)| \leq K(1+|x|)^{-\beta} \quad \text{for} \quad x \in R, \beta > 1. \tag{1.3}$$

Then it is possible to interchange sum and integral in (1.1)(see[1]). So we can write P_m as an integral operator

$$(P_m f)(x) = \int_{-\infty}^{\infty} 2^m q(2^m x, 2^m y) f(y) dy,$$

where the kernel $q(x,y)$ is defined by

$$q(x,y) = \sum_{n\in Z} \phi(x-n)\phi(y-n) \quad \text{for} \quad x,y \in R.$$

For $\beta > 1$, we also have $\int_{-\infty}^{\infty} q(x,y)dx = 1$ (see also[1],[3, p33]). For the trigonometric series the following fact is well known:

PROPOSITION [4, P57]. *Suppose f is integrable and 2π periodic, and of bounded variation in an interval I. If $f(x^+)$ and $f(x^-)$ exist at $x \in I$, then the Fourier series of f converges to $\frac{1}{2}\{f(x^+)+f(x^-)\}$. Moreover, if f is continuous on I, then the Fourier series converges uniformly on any closed subinterval of I.*

For wavelet expansions, Walter[2] has shown the following fact:

PROPOSITION. *Let $f \in L^1(R) \cap L^2(R)$, continuous on (a,b) and let f_m be the projection of f onto V_m, then*

$$f_m \to f \quad \text{as} \quad m \to \infty$$

uniformly on compact subsets of (a,b).

So it is natural to ask if wavelet expansions have the same property as the Fourier series has at point of discontinuity.

2. Wavelet expansions at discontinuity

In [2], it is assumed that $\phi(x) \in S_r, r \in N$, i.e., $|\phi^k(x)| \leq C_{pk}(1+|x|)^{-p}, k = 0, \cdots, r, p \in Z, x \in R$. But we don't assume any regularity condition on ϕ. In this paper we only assume the decay condition in (1.3).

LEMMA 1. *Let $p(x) = \int_x^\infty q(x,y)dy$ and $p_m(x) = \int_{2^m x}^\infty q(2^m x, y)dy$; then we have*

(i) $p(x)$ is 1-periodic function.

(ii) for $x = 2^{-k}j, j \in Z, p_m(x) = p(0)$ whenever $m \geq k$.

Proof. (i) follows by observing $q(x,y) = q(x+k, y+k), k \in Z$;

$$\begin{aligned} p(x+1) &= \int_{x+1}^\infty q(x+1,y)dy = \int_x^\infty q(x+1,s+1)ds \\ &= \int_x^\infty q(x,s)ds = p(x). \end{aligned}$$

For (ii), we have $p_m(x) = p(2^m x) = p(2^{m-k}j)$. If $m \geq k$, then $p_m(x) = p(\text{some integer})$. Hence $p_m(x) = p(0)$ by (i). □

THEOREM 1. *Suppose $f \in L^1(R) \cap L^\infty(R)$ is piecewise continuous. Let x be a dyadic rational, i.e., $x = 2^{-k}j$, for $j, k \in Z$; then*

$$(P_m f)(x) = f_m(x) \longrightarrow \alpha f(x^+) + (1-\alpha)f(x^-) \quad \text{as} \quad m \to \infty,$$

where $\alpha = \int_0^\infty q(0,y)dy$.

Proof. It can be proved by observing the orthogonal projection $P_m f$ of f onto V_m;

$$\begin{aligned}(P_m f)(x) &= \int_{-\infty}^{\infty} q_m(x,y)f(y)dy \\ &= \int_x^{\infty} q_m(x,y)f(y)dy + \int_{-\infty}^{x} q_m(x,y)f(y)dy \\ &= \int_{2^m x}^{\infty} q(2^m x, s)f(2^{-m}s)ds + \int_{-\infty}^{2^m x} q(2^m x, s)f(2^{-m}s)ds;\end{aligned}$$

by taking $x = 2^{-k}j$,

$$\begin{aligned}(P_m f)(x) = &\int_{2^{m-k}j}^{\infty} q(2^{m-k}j, s)f(2^{-m}s)ds \\ &+ \int_{-\infty}^{2^{m-k}j} q(2^{m-k}j, s)f(2^{-m}s)ds;\end{aligned}$$

by taking $y = s - 2^{m-k}j$ and $m \geq k$,

$$\begin{aligned}(P_m f)(x) &= \int_0^{\infty} q(2^{m-k}j, y + 2^{m-k}j)f(2^{-m}y + 2^{-k}j)dy \\ &\quad + \int_{-\infty}^{0} q(2^{m-k}j, y + 2^{m-k}j)f(2^{-m}y + 2^{-k}j)dy \\ &= \int_0^{\infty} q(0,y)f(2^{-m}y + 2^{-k}j)dy \\ &\quad + \int_{-\infty}^{0} q(0,y)f(2^{-m}y + 2^{-k}j)dy\end{aligned}$$

In the second equality, we have used Lemma 1. By (1.3), we have $|q(x,y)| \leq C(1+|x-y|)^{-\beta}, \beta > 1$ [1]. So by the Lebesgue dominated convergence theorem, we obtain as $m \to \infty$

$$\int_0^{\infty} q(0,y)f(x^+)dy + \int_{-\infty}^{0} q(0,y)f(x^-)dy = \alpha f(x^+) + (1-\alpha)f(x^-),$$

where $\alpha = \int_0^\infty q(0,y)dy = 1 - \int_{-\infty}^0 q(0,y)dy$. □

COROLLARY. *Suppose scaling function $\phi(x)$ is even; then*

$$(P_m f)(x) \longrightarrow \frac{1}{2}\{f(x^+) + f(x^-)\} \quad \text{as} \quad m \to \infty,$$
$$\text{for all dyadic rational} \quad x.$$

Proof. By the evenness of ϕ, we have

$$\begin{aligned} q(0,-y) &= \sum_{n\in Z} \phi(-n)\phi(-y-n) \\ &= \sum_{n\in Z} \phi(n)\phi(y+n) \\ &= q(0,y), \end{aligned}$$

and

$$\begin{aligned} 1 = \int_{-\infty}^{\infty} q(0,y)dy &= \int_0^{\infty} q(0,y)dy + \int_{-\infty}^{0} q(0,y)dy \\ &= \int_0^{\infty} q(0,y)dy + \int_0^{\infty} q(0,-y). \end{aligned}$$

Hence $2\int_0^{\infty} q(0,y)dy = 1$,and we take α as $\frac{1}{2}$ in Theroem 1. □

For an example, we show the wavelet expansion of the Haar system does not converge at non-dyadic rational.

EXAMPLE. Let ϕ be the scaling function for Haar wavelet. We consider a function f defined by

$$f(x) = \begin{cases} 0, & x \geq 2 \\ 1, & \frac{1}{3} \leq x < 2 \\ 0, & x < \frac{1}{3} \end{cases}.$$

Then,by using the notations in Lemma 1, we have

$$\begin{aligned} (P_m f)(\frac{1}{3}) &= \int_{\frac{1}{3}}^{2} 2^m q(\frac{2^m}{3}, 2^m y)dy \\ &= \int_{\frac{2^m}{3}}^{2^{m+1}} q(\frac{2^m}{3}, t)dt = p(\frac{2^m}{3}). \end{aligned}$$

Since $q(x,y) = \phi(y - [x]), [x] :=$ the greatest integer no bigger than x, we obtain

$$P_m(\frac{1}{3}) = p(\frac{2^m}{3}) = \int_{\frac{2^m}{3}}^{[\frac{2^m}{3}]+1} dt$$
$$= 1 + \left[\frac{2^m}{3}\right] - \frac{2^m}{3} = 1 - \left(\frac{2^m}{3} - \left[\frac{2^m}{3}\right]\right),$$

and

$$\begin{aligned} P_0(\frac{1}{3}) &= p(\frac{1}{3}) = 1 - \frac{1}{3} = \frac{2}{3} \\ P_1(\frac{1}{3}) &= p(\frac{2}{3}) = 1 - \frac{2}{3} = \frac{1}{3} \\ P_2(\frac{1}{3}) &= p(\frac{4}{3}) = 1 - (\frac{4}{3} - 1) = \frac{2}{3} \\ P_3(\frac{1}{3}) &= p(\frac{8}{3}) = 1 - (\frac{8}{3} - 2) = \frac{1}{3} \\ P_4(\frac{1}{3}) &= p(\frac{16}{3}) = 1 - (\frac{16}{3} - 5) = \frac{2}{3} \\ &\vdots \end{aligned},$$

which shows $(P_m f)(\frac{1}{3})$ does not converge as $m \to \infty$.

Even though the Shannon scaling function $\phi(x) = \frac{\sin \pi x}{\pi x}$ does not satisfy the decay condition in (1.3), it has the same property as the Fourier series does under an additional condition.

THEOREM 2. *Suppose ϕ is the scaling function associated with the Shannon wavelet. Let $f \in L^1(R) \cap L^2(R)$ be piecewise continuous and satisfy the Lipschitz condition to the left and right side of $x \in R$, then*

$$(P_m f)(x) \longrightarrow \frac{1}{2}\{f(x^+) + f(x^-)\} \quad \text{as} \quad n \to \infty.$$

Proof. From Lemma 1, for all $m \in Z$, we have

$$\begin{aligned}
\alpha_m(x) = p(2^m x) &= \int_{2^m x}^{\infty} q(2^m x, y)dy \\
&= \int_{2^m x}^{\infty} \frac{\sin \pi(2^m x - y)}{\pi(2^m x - y)} dy \\
&= \frac{1}{\pi} \int_0^{\infty} \frac{\sin t}{t} dt = \frac{1}{2} \quad .
\end{aligned}$$

Moreover,

$$\begin{aligned}
(P_m f)(x) &= \int_x^{\infty} q_m(x, y) f(y) dy + \int_{-\infty}^{x} q_m(x, y) f(y) dy \\
&= \int_{2^m x}^{\infty} q(2^m x, s) f(2^{-m} s) ds + \int_{-\infty}^{2^m x} q(2^m x, s) f(2^{-m} s) ds \\
&= \int_0^{\infty} q(2^m x, 2^m x + t) f(2^{-m} t + x) dt \\
&\qquad + \int_{-\infty}^{0} q(2^m x, 2^m x + t) f(2^{-m} t + x) dt \\
&= \int_0^{\infty} \frac{\sin \pi t}{\pi t} f(2^{-m} t + x) dt + \int_{-\infty}^{0} \frac{\sin \pi t}{\pi t} f(2^{-m} t + x) dt.
\end{aligned}$$

Now we consider the following calculation;

$$\begin{aligned}
(P_m f)(x) &- \frac{1}{2}\{f(x^+) + f(x^-)\} \\
&= \int_0^{\infty} \frac{\sin 2^m \pi t}{\pi t} \{f(t + x) - f(x^+)\} dt \\
&\quad + \int_{-\infty}^{0} \frac{\sin 2^m \pi t}{\pi t} \{f(t + x) - f(x^-)\} dt.
\end{aligned}$$

We show that the first integral converges to 0 as $m \to \infty$. Then the second integral has the same result by the same manner.

$$\begin{aligned}\int_0^\infty \frac{\sin 2^m \pi t}{\pi t}\{f(t+x)-f(x^+)\}dt &= \int_0^1 \sin 2^m\pi t\left\{\frac{f(t+x)-f(x^+)}{\pi t}\right\}dt \\ &+\int_1^\infty \sin 2^m\pi t\frac{f(t+x)}{\pi t}dt \\ &-f(x^+)\int_1^\infty \frac{\sin 2^m \pi t}{\pi t}dt \\ &= I_1+I_2+I_3.\end{aligned}$$

To finish the proof, we need to show $I_i \to 0$ as $m \to \infty$ for $i = 1, 2, 3$. Indeed we have the following estimate: by the Lipschitz condition for f, we get

$$\left|\frac{f(t+x)-f(x^+)}{\pi t}\right| \leq M, \quad \text{for a constant } M.$$

This tells $\frac{f(t+x)-f(x^+)}{\pi t} \in L^1(0,1)$ and by using the Riemann-Lebesgue lemma, we obtain $I_1 \to 0$ as $m \to \infty$. For I_2, since $\frac{f(t+x)}{t} \in L^1(1,\infty)$ we again obtain $I_2 \to 0$ as $m \to \infty$ by the Riemann -Lebesgue lemma. For I_3, direct calculation shows

$$I_3 = \left[-\frac{\cos 2^m\pi t}{2^m\pi}\frac{1}{t}\right]_1^\infty - \int_1^\infty \frac{\cos 2^m\pi t}{2^m\pi}\frac{1}{t^2}dt \quad ,$$

which converges to 0 as $m \to \infty$. □

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