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THE VOLUME/DIAMETER RATIO PINCHING SPHERE THEOREMS

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1. Introduction

In this paper, M will always denote an n- dimensional complete Riemannian manifold and ω_n is the volume of S^n.

Wu [Wu] showed that if $K(M) \geq 1$, then $\frac{vol(M)}{d(M)} \leq \frac{\omega_n}{\pi}$ and the equality holds iff M is isometric to S^n or P^n. Here we come across a natural question , that is , what can we say about M if $K(M) \geq 1$ and $\frac{vol(M)}{d(M)}$ is sufficiently close to $\frac{\omega_n}{\pi}$?

This paper is concerned with this problem and we established the following pinching theorem:

THEOREM A. *Given an integer $n \geq 2$, there exists $\varepsilon > 0$ such that if M is an n- dimensional Riemannian manifold with $K(M) \geq 1$, $\frac{\omega_n}{\pi} - \varepsilon \leq \frac{vol(M)}{d(M)} \leq \frac{\omega_n}{\pi}$, then M is diffeomorphic to S^n or P^n.*

In 1988, Durumeric proved the following theorem: Let (M, g_k) be a sequence of smooth Riemannian metrics with $1 \leq K(M, g_k) \leq K$, $d(M, g_k)$ converging to $\frac{\pi}{2}$ and $vol(M, g_k)$ converging to $\frac{\omega_n}{2}$. Then there exists a subsequnce converging to (P^n, can) in the Hausdorff-Lipschitz sense. Now we can see that the proof of Theorem A implies that the above theorem can be generalized.

Generally, under $Ric(M) \geq n-1$, the inequlity in the [Wu] is not true, but we may obtain some analogous results by pinching the excess of M. Recall that the excess of M is defined by $e(M) = \min_{p,q \in M} \max_{x \in M} e_{p,q}(x)$, where $e_{p,q}(x) = d(p,x) + d(q,x) - d(p,q)$.

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THEOREM B. *Given an integer $n \geq 2$, there exists $\varepsilon > 0$ such that if M is an n- dimensional Riemannian manifold with $n-1 \leq Ric(M) \leq C$, $\frac{\omega_n}{\pi} - \varepsilon \leq \frac{vol(M)}{d(M)}$ and $e(M) < \varepsilon$, then M is diffeomorphic to S^n.*

Recall that under $Ric(M) \geq n-1$, if $vol(M)$ is close to ω_n, then it follows immediately that $d(M)$ is close to π and $e(M)$ is close to zero. So in the above theorem, the volume/diameter and excess conditions can be repalced by volume condition

THEOREM C. *Given an integer $n \geq 2$, there exists $\varepsilon > 0$ such that if M is an n- dimensional Riemannian manifold with $Ric(M) \geq n-1, K(M) \geq k \in R, \frac{\omega_n}{\pi} - \varepsilon \leq \frac{vol(M)}{d(M)}$ and $e(M) < \varepsilon$, then M is diffeomorphic to S^n.*

By the proof of Theorem C, we have

THEOREM D. *Given an integer $n \geq 2$, there exists $\varepsilon > 0$ such that if M is an n- dimensional Riemannian manifold with $Ric(M) \geq n-1, \frac{\omega_n}{\pi} - \varepsilon \leq \frac{vol(M)}{d(M)}$ and $e(M) < \varepsilon$, then M is homeomorphic to S^n.*

Using C^α-compactness theorem in [AC], we easily obtain

THEOREM E. *Given an integer $n \geq 2$, there exists $\varepsilon > 0$ such that if M is an n- dimensional Riemannian manifold with $Ric(M) \geq n-1, inj(M) \geq \rho > 0, \frac{\omega_n}{\pi} - \varepsilon \leq \frac{vol(M)}{d(M)}$ and $e(M) < \varepsilon$, then M is diffeomorphic to S^n.*

2. Proof of the theorems

Throughout this section, we adopt the convention that if $p \in M$, then $B_p(r) = \{x \in M : d(x,p) < r\}$ and $\tilde{B}(r)$ is the metric r ball in S^n. Recall that the Bishop-Gromov volume comparison theorem says that if $Ric(M) \geq n-1$, then $vol(\tilde{B}(r)) \geq vol(B_p(r))$ for each r.

Proof of Theorem A. Suppose not! Then there exists a sequence of manifolds M_i such that $K(M_i) \geq 1, \frac{vol(M_i)}{d(M_i)} \to \frac{\omega}{\pi}$ and M_i is neither diffeomorphic to S^n nor P^n. By passing to a subseqquece ,if necessary, we may assume $d(M_i) \leq \frac{\pi}{2}$ for all i or $d(M_i) > \frac{\pi}{2}$ for all i.

CASE 1. $d(M_i) \leq \frac{\pi}{2}$ for all i : Note that there exists a space M such that $M_i \to M$ in the Gromov-Haussdorff topology. let $l = d(M)$ Then $d(M_i) =: l_i \to l \leq \frac{\pi}{2}$ and we have $vol(M_i) \to \frac{\omega_n}{\pi} l$, or

$$\frac{vol(M_i)}{\omega_n} \leq \frac{\int_0^{l_i} \sin^{n-1} r dr}{\int_0^{\pi} \sin^{n-1} r dr} \leq \frac{l_i}{\pi} \to \frac{l}{\pi}.$$

Thus

$$\frac{\int_0^{l} \sin^{n-1} r dr}{l} = \frac{\int_0^{\pi} \sin^{n-1} r dr}{\pi}.$$

Now since $f(x) = \frac{\int_0^x \sin^{n-1} r dr}{x}$ is strictly increasing function of $x (\leq \frac{\pi}{2})$, we have $l = \frac{\pi}{2}$ and $vol(M_i) \to \frac{\omega_n}{2}$. Now by [OSY], we have M_i is diffeomorphic to P^n for large i ,which is a contradiction.

CASE 2. $d(M_i) > \frac{\pi}{2}$ for all i : In Lemma 3.2 of [GP], we have the following: Suppose $M_i \to X$ in the Gromov-Haussdorff topology and $K(M_i) \geq 1, d(M_i) = D \in [\frac{\pi}{2}, \pi], vol(M_i) \to \frac{D}{\pi} vol(S^n)$ and $M_i \neq P^n$, then D must be π. The proof of this relies essentially on the volume limit argument and a slight modification of this proof makes it possible to say that "$d(M_i) = D$" above can be replaced by "$d(M_i) \to D$". Consequently, we have in our situation that $vol(M_i) \to vol(S^n)$ an dM_i is diffeomorphic to S^n for large i ,which is a contradiction. □

Let M be a complete Riemannian manifold with $Ric(M) \geq n - 1, e(M) = 0$. Then it is easy to see that $M = \bar{B}_p(\alpha) \cup \bar{B}_p(\beta)$ for any $\alpha > 0, \beta > 0$ with $\alpha + \beta = d(M)$ and $e_{p,q}(x) = 0$ for every $x \in M$. Note that $d(M) = d(p, q)$, since $d(M) \leq e(M) + d(p, q)$, if $e(M) = \max_x e_{p,q}(x)$. Similarly, if M is a complete Riemannian manifold with $Ric(M) \geq n - 1, e(M) < \varepsilon$, then we have $M = \bar{B}_p(\alpha + \frac{\varepsilon}{2}) \cup \bar{B}_p(\beta + \frac{\varepsilon}{2})$, where $e(M) = \max_x e_{p,q}(x)$ and $\alpha + \beta = d(p, q)$.

Proof of Theorem B. Consider a sequence of manifolds M_i such that $n - 1 \leq Ric(M_i) \leq C, \frac{\omega_n}{\pi} - \varepsilon_i \leq \frac{vol(M_i)}{d(M_i)}$ and $e(M_i) < \varepsilon_i$,where $\lim_{i \to \infty} \varepsilon_i = 0, \varepsilon_i > 0$. Passing to a subsequence ,if necessary, we assume that $d(M_i) \leq \frac{\pi}{2}$ for all i or $d(M_i) > \frac{\pi}{2}$ for all i.

CASE 1. $d(M_i) > \frac{\pi}{2}$ for all i : Let p_i, q_i be the points satisfying $\max_x e_{p_i, q_i}(x) = e(M_i)$, and put $d_i = d(p_i, q_i)$. From $d(M_i) \leq e(M_i) +$

$d(p_i, q_i)$, it follow immediately that $d := \lim_{i\to\infty} d_i = \lim_{i\to\infty} d(M_i) \geq \frac{\pi}{2}$. So we can choose α_i, β_i so that $\alpha_i + \beta_i = d_i$ and $\alpha_i \uparrow \frac{\pi}{2}$, $\beta_i \uparrow d\frac{\pi}{2}$. Now we have

$$\begin{aligned}
\frac{vol(M_i)}{d(M_i)} &\leq \frac{1}{d_i}\left\{vol(B_{p_i}(\alpha_i + \frac{\epsilon_i}{2})) + vol(B_{q_i}(\beta_i + \frac{\epsilon_i}{2}))\right\} \\
&\leq \frac{1}{d_i}\left\{vol(\tilde{B}(\alpha_i + \frac{\epsilon_i}{2})) + vol(\tilde{B}(\beta_i + \frac{\epsilon_i}{2}))\right\} \\
&= \frac{1}{d_i}\left\{\omega_{n-1}\int_0^{\alpha_i+\frac{\epsilon_i}{2}} \sin^{n-1} t dt + \omega_{n-1}\int_0^{\beta_i+\frac{\epsilon_i}{2}} \sin^{n-1} t dt\right\} \\
&= \frac{\omega_{n-1}}{d_i}\left\{\int_0^{\alpha_i} \sin^{n-1} t dt + \int_0^{\beta_i} \sin^{n-1} t dt\right\} + \delta_i \\
&\leq \frac{\omega_{n-1}}{d_i}\left\{\frac{\alpha_i}{\pi}\int_0^{\pi} \sin^{n-1} t dt + \frac{\beta_i}{\pi}\int_0^{\pi} \sin^{n-1} t dt\right\} + \delta_i \\
&= \frac{\omega_n}{\pi} + \delta_i,
\end{aligned}$$

where $\delta_i \to 0$ as $i \to \infty$. Since $\frac{vol(M_i)}{d(M_i)} \to \frac{\omega_n}{\pi}$, we obtain by letting $i \to \infty$, that $\frac{1}{d-\pi/2}\int_0^{d-\pi/2} \sin^{n-1} t dt = \frac{1}{\pi}\int_0^{\pi} \sin^{n-1} t dt$. Consequently, $d - \pi/2 = \pi/2$ or $d = \pi$. So $vol(M_i) \to \omega_n$ and the result follows by [AC].

CASE 2. $d(M_i) \leq \frac{\pi}{2}$ for all i : In this case, we observed that $d(M_i) \to \frac{\pi}{2}$ previously. Under the same setting as in case 1, choose α_i, β_i so that $\alpha_i \uparrow \pi/3$,$\beta_i \uparrow \pi/6$. Then we have

$$\begin{aligned}
\frac{vol(M_i)}{d(M_i)} &\leq \frac{\omega_{n-1}}{d_i}\left\{\int_0^{\alpha_i} \sin^{n-1} t dt + \int_0^{\beta_i} \sin^{n-1} t dt\right\} + \delta_i \\
&\leq \frac{\omega_{n-1}}{d_i}\left\{\frac{\alpha_i}{\pi}\int_0^{\pi} \sin^{n-1} t dt + \frac{\beta_i}{\pi}\int_0^{\pi} \sin^{n-1} t dt\right\} + \delta_i \\
&= \frac{\omega_n}{\pi} + \delta_i.
\end{aligned}$$

By letting $i \to \infty$, we have a contradiction to the strict increasing property of $f(x) = \frac{\int_0^x \sin^{n-1} r dr}{x} (0 \leq x \leq \frac{\pi}{2})$ $\square$

Proof of Theorem C. The result can be easily obtained by the same argument as theorem B together with theorem A in [Y] which states that any complete Riemannian n-manifold with $Ric(M) \geq n-1, K(M) \geq k \in R$ and $vol(M)$ close to ω_n is isometric to S^n. □

Proof of Theorem D. Recall that Perelman [P] showed that any complete $n-$ manifold with $Ric(M) \geq n-1$ and $vol(M)$ close to ω_n is homeomorphic to S^n. This result together with the proof of theorem C says the result is true. □

LEMMA. *Let M be an n-dimensional complete Riemannian manifold with $Ric(M) \geq n-1, e(M) = 0$. Then $vol(M)/d(M) \leq \frac{\omega_n}{\pi}$ and the equality holds iff M is isometric to S^n.*

Proof. CASE 1. $d(M) \leq \frac{\pi}{2}$: In this case, $vol(M)/d(M) \leq \frac{\omega_n}{\pi}$ follows from the same argument in Theorem A, and we observed that $vol(M)/d(M) = \frac{\omega_n}{\pi}$ implies $d(M) = \frac{\pi}{2}$. So

$$\begin{aligned}
\frac{\omega_n}{\pi} = \frac{vol(M)}{d(M)} &\leq \frac{1}{\pi/2}\left\{vol(\tilde{B}(\frac{\pi}{6})) + vol(\tilde{B}(\frac{\pi}{3}))\right\} \\
&\leq \frac{2}{\pi}\omega_n\left\{\int_0^{\frac{\pi}{6}} \sin^{n-1} tdt + \int_0^{\frac{\pi}{3}} \sin^{n-1} tdt\right\} \\
&< \frac{2}{\pi}\omega_n\left\{\frac{\pi/6}{\pi}\int_0^{\frac{\pi}{6}} \sin^{n-1} tdt + \frac{\pi/3}{\pi}\int_0^{\frac{\pi}{3}} \sin^{n-1} tdt\right\} \\
&= \frac{\omega_n}{\pi},
\end{aligned}$$

which is a contradiction.

CASE 2. $d(M) > \frac{\pi}{2}$: Let $d = d(M)$. Then

$$\begin{aligned}
\frac{vol(M)}{d(M)} &= \frac{1}{d}\left\{vol(\tilde{B}(\frac{\pi}{2})) + vol(\tilde{B}(d - \frac{\pi}{2}))\right\} \\
&\leq \frac{1}{d}\omega_n\left\{\int_0^{\frac{\pi}{2}} \sin^{n-1} tdt + \int_0^{d-\frac{\pi}{2}} \sin^{n-1} tdt\right\} \\
&\leq \frac{\omega_n}{d}\left\{\frac{\pi/2}{\pi}\int_0^{\pi} \sin^{n-1} tdt + \frac{d-\pi/2}{\pi}\int_0^{\pi} \sin^{n-1} tdt\right\} \\
&= \frac{\omega_n}{\pi}.
\end{aligned}$$

Thus the equality holds iff $d - \frac{\pi}{2} = \frac{\pi}{2}$, that is, $d = \pi$. So M is isometric to S^n, by the Cheng's maximal diameter theorem, which states that any complete Riemannian manifold with $Ric(M) \leq n - 1$ and $d(M) = \pi$ is isometric to S^n. □

Proof of Theorem E. This is an immediate consequence of the above lemma and C^α -compactness theorem in [AC], which states that given any sequence of compact Riemannian manifolds (M_i, g_i) such that Ric $(M_i) \geq n-1, inj(M_i) \geq \rho > 0, vol(M) \leq V$, and given any fixed $\alpha < 1$, there is a compact manifold M, and diffeomorphisms $f_j : M \to M_j$, for a subsequence $\{j\}$ of $\{i\}$, such that the metrics $f_j^* g_j$ converge in the C^β topology for $\beta < \alpha$, to a Riemannian manifold (M, g) with C^α metric g. □

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