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# INVERSE PROBLEM FOR SEMILINEAR CONTROL SYSTEMS

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## 1. Introduction

Let consider the following problem: find an element $u(t)$ in a Banach space $U$ from the equation

$$x'(t) = Ax(t) + f(t, x(t)) + \Phi_0 u(t), \quad 0 \le t \le T$$

with initial and terminal conditions

$$x(0) = 0, \quad x(T) = \phi$$

in a Banach space $X$ where $\phi \in D(A)$. This problem is a kind of control engineering inverse problem and contains nonlinear term, so that it is difficult and interesting. The proof of main result in this paper is based on the Fredholm property of [1] in section 3. Similar considerations of linear system have been dealt with in many references. Among these literatures, Suzuki[5] introduced this problem for heat equation with unknown spatially-varing conductivity. Nakagiri and Yamamoto[2] considered the identifiability problem, which $A$ is a unknown operator to be identified, where the system is described by a linear retarded functional differential equation. We can also apply to determining the magnitude of the control set for approximate controllability if $X$ is a reflexive space, i.e., we can consider whether a dense subset of $X$ is covered by reachable set in section 4.

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## 2. Preliminaries

Let both $X$ and $U$ be Banach spaces. Consider the following semilinear equation which is described by control system with final time on $X$:

$$
\text{(2.1)} \qquad \begin{cases} x'(t) = Ax(t) + f(t, x(t)) + \Phi_0 u(t), & 0 \le t \le T, \\ x(0) = 0 \\ x(T) = \phi \in D(A). \end{cases}
$$

Here, the operator $A$ that generates a compact $C_0$-semigroup $S(t)$ on $X$ is a bounded operator. For the sake of simplicity we assume that the complex spectrum of $A$ is contained in the half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$, and hence there exists a constant $M$ such that

$$
\text{(2.2)} \qquad ||S(t)|| \le M
$$

where the norm of an element of $X$ is usually denoted by $|| \cdot ||$. Let us assume that $f$ is continuously differentiable on $[0, T] \times X$ into $X$ and $f(0, x(0)) = 0$. Let $\Phi_0$ be a bounded operator from $U$ onto $X$ and assume that there exists a constant $c$ such that

$$
||u|| \le c||\Phi_0 u||, \quad u \in U.
$$

Then the initial value problem (2.1) has a unique solution satisfying the integral equation

$$
\text{(2.3)} \qquad x(t) = \int_0^t S(t-s)\{f(s, x(s)) + \Phi_0 u(s)\}ds,
$$

for $0 \le t \le T$. This solution $x(t)$ is continuously differentiable in $X$ on $[0, T]$, $x(t) \in D(A)$ and (2.1) is satisfied in $X$(see [3, Theorem 1.5 in Chapter 6]). Let $x(t; f, u)$ be a solution of the equation (2.1) associated with nonlinear term $f$ and control $u$ at time $t$. We define the reachable sets for the equation (2.1) as follows:

$$
\begin{aligned}
L_t &= \{x(t; 0, u) : u \in C^1(0, T; U)\}, \\
R_t &= \{x(t; f, u) : u \in C^1(0, T; U)\}, \\
R_\infty &= \bigcup_{t \ge 0} R_t,
\end{aligned}
$$

where $C^1(0,T;U)$ denotes the set of all continuously differentiable functions from $(0,T)$ into a Banach space $U$.

Let $X$ be a reflexive Banach space. The observed linear system on $X^*$ is defined by

$$(2.4)\qquad \begin{cases} y'(t) = A^*y(t), & 0 \le t \le T, \\ y(0) = 0. \end{cases}$$

Then we can define the observable sets by

$$N_t = \{x^* \in X^* : \Phi_0^* S^*(s)x^* = 0, \quad s \in [0,t]\}$$
$$N_\infty = \bigcap_{t \ge 0} N_t.$$

For the solvability of the equation (2.1) we will use properties of degree theory, which is called Fredholm alternative for nonlinear operator.

Suppose that $D$ is open subset of $X$, $p$ is a point in $X$ and $B$ is a continuous function from $\overline{D}$ into $X$ where $\overline{D}$ denotes the closure of $D$ in $X$. Set

$$d(B; D, p) = \sum_{x \in B^{-1}(p)} \operatorname{sign} J(B(x))$$

$J(B(x))$ is the Jacobian determinant of $B$ at $x \in D$. Then $d(B; D, p)$ is called the degree of the mapping $B$ with respect to the set $D$ and the point $p$. Let $B$ be a compact operator. Then if $d(I - B; D, 0) \neq 0$ and $(I - B)(x) \neq 0$ for all $x \in \partial D$ then there is an $x_0 \in D$ such that $(I - B)(x_0) = 0$, which is called the existence theorem and this is the aim of using degree theory in this paper. Let $B(t)$ be a compact operator from $\overline{D}$ into $X$. Suppose that if given $\epsilon > 0$, there exists a $\delta > 0$ such that if $|t_1 - t_2| < \delta$ then for all $x \in D$ it is

$$||B(t_1)(x) - B(t_2)(x)|| < \epsilon.$$

It is well known that if for all $x \in \partial D$ and all $t \in [0,1]$, $(I - B(t))(x) \neq 0$, then $d(I - B(t); D, 0)$ exists and has the same value, which is called the invariance under homotopy theorem.

## 3. Inverse problem

In what follows we consider the control problem corresponding to the admissible set $W$ defined by

$$W = \{u \in C^1(0,T;U) : u(0) = a,\ a \in U\}. \tag{3.1}$$

So, we can define the reachable set by

$$L_t^{'} = \{x(t;0,u) : u \in W\},$$
$$R_t^{'} = \{x(t;f,u) : u \in W\}.$$

corresponding to the admissible set $W$.

In (2.1), for each $u \in L^2(0,T;U)$ if $f$ is continuously differentiabe in $t$ on $[0,T]$ and uniformly Lipschitz continuous on $X$ then there exists a unique mild soluion $x(t;f,u)$ satisfying (2.3). So we can define the nonlinear operator $F$ on $W$ by

$$(Fu)(t) = f(x(t)), \quad u \in W$$

and we set $(\Phi u)(t) = \Phi_0 u(t)$. This operator $\Phi$ is called the Nemitsky superposition operator corresponding to $\Phi_0$. Both $F$ and $\Phi$ are continuous operators from $W$ to $X$ and $\Phi$ is a bounded linear operator. Since the semigroup $S(t)$ is compact, the assumption $f$ implies the compactness of the operator $F$. We note that since the controller $\Phi$ is bounded below there exists the bounded inverse $\Phi^{-1}$.

According to the definition $x(t;f,u)$, we have

$$x^{'}(T) = S(T)\Phi_0 a + \int_0^T S(T-s)D(s)((F+\Phi)u)(s)ds \tag{3.2}$$

where we set $D(s) = d/ds$. From (2.1), (3.2) and $x(T) = \phi \in D(A)$ it follows

$$\begin{aligned} &A\phi + \Phi_0 u(T) + f(T,\phi) \\ &\quad = S(T)\Phi_0 a + \int_0^T S(T-s)D(s)((F+\Phi)u)(s)ds. \end{aligned}$$

Define the operator $B$ from $W$ to $X$ by

$$Bu = \Phi^{-1}[\int_0^T S(T-s)(D(s)((F+\Phi)u)(s))]ds.$$

LEMMA 3.1. *The operator $B$ mentioned above is compact.*

*Proof.* Let

$$G(T) = \int_0^T S(T-s)(D(s)((F+\Phi)\cdot))(s)ds.$$

Then we can rewrite $G(T)$ as

$$G(T) = S(\epsilon)G(T-\epsilon) + \int_{T-\epsilon}^{T} S(T-s)(D(s)((F+\Phi)\cdot))(s)ds$$

for $\epsilon \in (0,T]$. From the assumptions $f$ and $\Phi$ it follows

$$\|\int_{T-\epsilon}^{T} S(T-s)(D(s)((F+\Phi)\cdot))(s)ds\| \leq$$

$$(\sup_{s\in[0,T]} \|(D(s)(F+\Phi)\cdot))(s))\| \int_0^{\epsilon} \|S(s)\|ds$$

tends zero as $\epsilon \longrightarrow 0$. By the compactness of $S(\epsilon)$, $\Phi^{-1}S(\epsilon)G(T-\epsilon)$ is compact and hence, $B$ is also compact as a limit of compact operator.

LEMMA 3.2. *There exists a constant $c_0 > 0$ such that*

$$\|Bu\| \leq c_0\|u\|_{C^1} \tag{3.3}$$

*where $c_0$ is depend on time $T$.*

*Proof.* We have that continuous differentiable of $f$ implies that $f$ is Lipschitz continuous in both $t$ and $x$, and by assumption that $f(0,x(0)) = 0$ it follows

$$\|(Fu)(t)\| = \|f(t,x(t))\| \leq c_1\|x(t)\|$$

where $c_1$ is the Lipschitz constant. Thus from (2.2) and (2.3)

$$\|x(t)\| \leq M\int_0^t \|(f(s,x(s)) + \Phi_0 u(s))\|ds$$
$$\leq c_1 M\int_0^t \|x(s)\|ds + MT\|\Phi_0\|\|u\|_{C^1}$$

for $0 < t \leq T$. By Gronwall's inequality we have

$$||x(t)|| \leq MT||\Phi_0||exp(c_1 MT)||u||,$$

hence

$$||(Fu)(t)|| \leq c_1 MT||\Phi_0||exp(c_1 MT)||u||.$$

From the similar way mentioned above and the uniformly Lipschitz continuity in both $x$ and $t$, since

$$D(s)((F+\Phi)u)(s) = \frac{\partial}{\partial x}f(s,x)x'(s) + \frac{\partial}{\partial s}f(s,x) + \Phi_0 u'(s)$$

we have

$$||D(s)((F+\Phi)u)(s)|| \leq c_2$$

for some $c_2 > 0$. Therefore, by the definition of the operator $B$, the proof is complete.

As is seen in Lemma 3.2, we remark it holds that $1 - c_0 > 0$ for the sufficiently small time $T$. In virtue of Lemmas in this section we know that $B$ is a compact operator, therefore from (3.2) we can reduce to the equation

$$(I - B)u = g, \quad g = \Phi^{-1}[S(T)\Phi_0 a - A\phi - f(T,\phi)].$$

THEOREM 3.1. *Let $||a|| < c_0$ where $a \in U$ in (3.1) and $1 - c_0 > 0$ where $c_0$ is a constant in (3.3). Then under the assumptions $f$ and $\Phi_0$, there exists a solution $u \in W$ of problem (2.1) for a given $\phi \in D(A)$.*

*Proof.* Consider the following equation on $W$:

$$(I - \lambda B)u = g, \quad 0 \leq \lambda \leq 1.$$

Take a constant $c > 0$ such that

$$c > (1 - c_0)^{-1}||g||.$$

We apply the degree theory on the ball $U_c$, i.e.,

$$U_c = \{u \in W : ||u|| < c\}.$$

Then, by using Lamma 3.2, we have

$$||u|| \leq ||g|| + ||Bu|| \leq ||g|| + c_0||u||.$$

Thus

$$||u|| \leq (1 - c_0)^{-1}||g|| < c,$$

that is,

$$u \notin \partial U_c, \quad 0 \leq \lambda \leq 1$$

where $\partial U_c$ denotes the boundary of $U_c$. Therefore, from degree theory it follows that there exists a solution $u \in U_c$ such that

$$(I - B)u = g.$$

## 4. Reachable sets for semilinear system

Let $X$ be a reflexive Banach space throughout this section.

DEFINITION 4.1. (1) The system (2.1) is said to be approximate controllable in time $T$ (resp. infinite time) if $\overline{R_T} = X$ (resp. $\overline{R_\infty} = X$). (2) The system (2.4) is said to be observable on $[0, T]$ (resp. infinite time) if $N_T = \{0\}$ (resp. $N_\infty = \{0\}$).

THEOREM 4.1. *Suppose that $1 - c_0 > 0$ where $c_0$ is a constant in (3.3) for sufficiently small $T$, then the system (2.1) is approximate controllable in time $T$.*

*Furthermore, the system (2.1) is approximate controllable in time $T$ if and only if the system (2.4) is observable in time $T$.*

*Proof.* As is seen in Theorem 3.1 it is easily seen that

$$R_T = L_T = D(A).$$

Since

$$L_T = \{ \int_0^T S(T - s)\Phi_0 u(s) ds : u \in C^1(0, T; U)\},$$

by the orthogonal set $L_T^{\perp}$ of $L_T$ we have

$$\begin{aligned} L_T^{\perp} &= \{\int_0^T S(s)\Phi_0 u(s)ds : u \in C^1(0,T;U)\}^{\perp} \\ &= ker\Phi_0^* S^*(t), \quad t \in [0,T] \\ &= N_T. \end{aligned}$$

Hence, from the definition 4.1 we have that the system (2.1) is approximate controllable in time $T$ if and only if the system (2.4) is observable in time $T$.

In what follows we assume that the control set $U$ is a reflexive Banach space and $u \in W^{2,p}(0,T;U)$ for $p > 1$ where

$$W^{2,p}(0,T;U) = \{u \in L^p(0,T;U) : u', u'' \in L^p(0,T;U)\}.$$

DEFINITION 4.2. (1) The system (2.1) is said to be exact contollable in time $T$ (resp. infinite time) if $R_T = X$ (resp. $R_\infty = X$).
(2) The system (2.4) is said to be continuously observable on $[0,T]$ if there exists a constant $M_T > 0$ such that

$$||f||_* \le M_T ||\Phi^* S^*(\cdot)f||_{(W^{2,p})_*} ds$$

for each $f \in X^*$.

THEOREM 4.2. *Let $D(A) = X$ and assume the assumptions in Theorem 4.1. Then the system is exact controllable in time $T$.*

*Furthermore, the system (2.1) is exact controllable in time $T$ if and only if the system (2.4) is continuously observable on $[0,T]$.*

*Proof.* Define $G : W^{2,p}(0,T;U) \longrightarrow X$ by

$$Gu = \int_0^T S(T-s)\Phi_0 u(s)ds, \quad u \in W^{2,p}(0,T;U),$$

then from the assumptions and Theorem 3.1 it follows that $R_T = L_T = X$. Hence, since $W^{2,p}(0,T;U)$ is reflexive it is equivalent to the fact $\operatorname{Im} G \supset \operatorname{Im} I$ where $I$ is the identity operator. Therefore there exists a constant $M_T$ such that

$$||f||_* \le M_T ||\Phi^* S^*(\cdot)f||_{(W^{2,p})_*} ds$$

for each $f \in X^*$. The proof of theorem is complete.

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