Bull. Korean Math. Soc. **33** (1996), No. 4, pp. 593–601

ON FUZZY IDEALS OF NEAR–RINGS

SEUNG DONG KIM AND HEE SIK KIM

Dedicated to Prof. Yoo Bong Chun for his 65th birthday

1. Introduction

W. Liu [11] has studied fuzzy ideals of a ring, and many researchers [5, 6, 7, 16] are engaged in extending the concepts. The notion of fuzzy ideals and its properties were applied to various areas: semigroups [8, 9, 10, 13, 15], distributive lattices [2], artinian rings [12], BCK-algebras [14], near-rings [1]. In this paper we obtained an exact analogue of fuzzy ideals for near-ring which was discussed in [5, 11].

A non-empty set R with two binary operations '+' and '·' is called a *near-ring* ([3]) if

(1) $(R, +)$ is a group,

(2) $(R, \cdot)$ is a semigroup,

(3) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

We will use the word 'near-ring' to mean 'left near-ring'. We denote xy instead of $x \cdot y$. Note that $x0 = 0$ and $x(-y) = -xy$ but in general $0x \neq 0$ for some $x \in R$. An *ideal* I of a near-ring R is a subset of R such that

(4) $(I, +)$ is a normal subgroup of $(R, +)$,

(5) $RI \subseteq I$,

(6) $(r + i)s - rs \in I$ for any $i \in I$ and any $r, s \in R$. Note that I is a *left ideal* of R if I satisfies (4) and (5), and I is a *right ideal* of R if I satisfies (4) and (6).

Received April 18, 1996.

1991 AMS Subject Classification: 03E72, 16Y30.

Key words and phrases: fuzzy ideal, near-ring, level subset, sup property.

2. Fuzzy ideals of near-rings

Let R be a near-ring and μ be a fuzzy subset of R. We say μ a *fuzzy subnear-ring* of R if

(7) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

(8) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in R$. μ is called a *fuzzy ideal* of R if μ is a fuzzy subnear-ring of R and

(9) $\mu(x) = \mu(y + x - y)$,

(10) $\mu(xy) \geq \mu(y)$,

(11) $\mu((x+i)y - xy) \geq \mu(i)$, for any $x, y, i \in R$. Note that μ is a fuzzy left ideal of R if it satisfies (7), (9) and (10), and μ is a fuzzy right ideal of R if it satisfies (7), (8), (9) and (11). (see [1])

We give some examples of fuzzy ideals of near-rings.

EXAMPLE 2.1. Let $R := \{a, b, c, d\}$ be a set with two binary operations as follows:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	b

Then we can easily see that $(R; +, \cdot)$ is a (left) near-ring. Define a fuzzy subset $\mu : R \to [0,1]$ by $\mu(c) = \mu(d) < \mu(b) < \mu(a)$. Then μ is a fuzzy ideal of R.

EXAMPLE 2.2. Let $R := \{a, b, c, d\}$ be a set with two binary operations as follows:

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	b	a
d	d	c	a	b

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	b	c	b

Then we can easily see that $(R; +, \cdot)$ is a (left) near-ring. Define a fuzzy subset $\mu : R \to [0,1]$ by $\mu(c) = \mu(d) < \mu(b) < \mu(a)$. Then μ is a fuzzy left ideal of R, but not fuzzy right ideal of R, since $\mu((c+b)d - cd) = \mu(d) < \mu(b)$.

LEMMA 2.3. *If a fuzzy subset μ of R satisfies the property (7) then*
(i) $\mu(0_R) \geq \mu(x)$,
(ii) $\mu(-x) = \mu(x)$, *for all* $x, y \in R$.

Proof. (i) We have that for any $x \in R$,

$$\mu(0_R) = \mu(x - x) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

(ii) By (i), we have that

$$\mu(-x) = \mu(0_R - x) \geq \min\{\mu(0_M), \mu(x)\} = \mu(x)$$

for all $x \in R$. Since x is arbitrary, we conclude that $\mu(-x) = \mu(x)$. □

PROPOSITION 2.4. *Let μ be a fuzzy ideal of R. If $\mu(x - y) = \mu(0_R)$ then $\mu(x) = \mu(y)$.*

Proof. Assume that $\mu(x - y) = \mu(0_R)$ for all $x, y \in R$. Then

$$\begin{aligned}\mu(x) &= \mu(x - y + y) \\ &\geq \min\{\mu(x - y), \mu(y)\} \\ &= \min\{\mu(0_R), \mu(y)\} \\ &= \mu(y).\end{aligned}$$

Similarly, using $\mu(y - x) = \mu(x - y) = \mu(0_R)$, we have $\mu(y) \geq \mu(x)$. □

P. S. Das [4] obtained a similar characterization of all fuzzy subgroups of finite cyclic groups by introducing the concept of level subsets. Z. Yue [16] and V. N. Dixit et al. [5] applied same idea to rings. We now apply this concept to near-rings. Let μ be a fuzzy subset of a near-ring R. For $t \in [0, 1]$, the set $\mu_t := \{x \in R | \mu(x) \geq t\}$ is called a *level subset* of the fuzzy subset μ.

THEOREM 2.5 [1]. *Let μ be a fuzzy subset of a near-ring R. Then the level subset μ_t is a subnear-ring (or ideal) of R for all $t \in [0, 1], t \leq \mu(0)$ if and only if μ is a fuzzy subnear-ring (or ideal), respectively.*

THEOREM 2.6. *Let I be a left (right) ideal of a near-ring R. Then for any $t \in (0,1)$, there exists a fuzzy left (right) ideal μ of R such that $\mu_t = I$.*

Proof. Let $\mu : R \to [0,1]$ be a fuzzy set defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in I, \\ 0 & \text{if } x \notin I, \end{cases}$$

where t is a fixed number in $(0,1)$. Then clearly $\mu_t = I$. Let $x, y \in R$. Then by routine calculations, we have that

$$\mu(x-y) \geq \min\{\mu(x), \mu(y)\}.$$

Assume that $\mu(x) < \mu(y+x-y)$ for some $x, y \in R$. Since μ is two-valued, i.e., $|Im(\mu)| = 2$, $\mu(x) = 0$ and $\mu(y+x-y) = t$ and hence $x \notin I$, $y+x-y \in I$. Since $(I,+)$ is a normal subgroup of $(R,+)$, $x = -x+(y+x-y)+x \in I$, a contradiction. Similarly, the assumption that $\mu(y+x-y) < \mu(x)$ also leads to a contradiction. We can easily see that $\mu(xy) \geq \mu(y)$ for any $x, y \in R$.

Suppose that I is a right ideal of R and assume $\mu((x+i)y-xy) < \mu(i)$ for some $x, y \in R$ and $i \in I$. Since $|Im(\mu)| = 2$, $\mu((x+i)y-xy) = 0$ and $\mu(i) = t$ and hence $(x+i)y-xy \notin I$ and $i \in I$, which leads to a contradiction, since I is a right ideal of R. This proves the theorem. □

THEOREM 2.7. *Let μ be a fuzzy left (right) ideal of a near-ring R. Then two level left (right) ideals μ_{t_1} and μ_{t_2} (with $t_1 < t_2$) of μ are equal if and only if there is no $x \in R$ such that $t_1 \leq \mu(x) < t_2$.*

Proof. ($\Rightarrow$) Suppose $t_1 < t_2$ and $\mu_{t_1} = \mu_{t_2}$. If there exists $x \in R$ such that $t_1 \leq \mu(x) < t_2$, then μ_{t_2} is a proper subset of μ_{t_1}. This is a contradiction.

($\Leftarrow$) Assume that there is no $x \in R$ such that $t_1 \leq \mu(x) < t_2$. From $t_1 < t_2$ it follows that $\mu_{t_2} \subseteq \mu_{t_1}$. If $x \in \mu_{t_1}$, then $\mu(x) \geq t_1$ and so $\mu(x) \geq t_2$ because $\mu(x) \not< t_2$. Hence $x \in \mu_{t_2}$. This completes the proof. □

THEOREM 2.8. *Let R be a near-ring and μ a fuzzy left (right) ideal of R. If $Im(\mu) = \{t_1, ..., t_n\}$, where $t_1 < ... < t_n$, then the family of left (right) ideals $\mu_{t_i} (i = 1, ..., n)$ constitutes all the level left (right) ideals of μ.*

Proof. Let $t \in [0,1]$ and $t \notin Im(\mu)$. If $t < t_1$, then $\mu_{t_1} \subseteq \mu_t$. Since $\mu_{t_1} = R$, it follows that $\mu_t = R$, so that $\mu_t = \mu_{t_1}$. If $t_i < t < t_{i+1} (1 \leq i \leq n-1)$ then there is no $x \in R$ such that $t \leq \mu(x) < t_{i+1}$. From Theorem 2.7, we have that $\mu_t = \mu_{t_{i+1}}$. This shows that for any $t \in [0,1]$ with $t \leq \mu(0_R)$, the level left ideal μ_t is in $\{\mu_{t_i} | 1 \leq i \leq n\}$. □

THEOREM 2.9. *Let I be a non-empty subset of a near-ring R and let μ be a fuzzy set in R such that μ is into $\{0,1\}$, so that μ is the characteristic function of I. Then μ is a fuzzy left (right) ideal of R if and only if I is a left (right) ideal of R.*

Proof. Assume that μ is a fuzzy left ideal of R. Let $x, y \in I$. Then $\mu(x) = \mu(y) = 1$. Thus $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} = 1$ and so $\mu(x - y) = 1$. This means that $x - y \in I$. Therefore I is an additive subgroup of R. Let $x \in R$ and $y \in I$. Then $\mu(xy) \geq \mu(y) = 1$ and hence $\mu(xy) = 1$. So $xy \in I$, and hence I is a left ideal of R. Assume that μ is a fuzzy right ideal of R. If $x, y \in R$ and $i \in I$, then $\mu((x + i)y - xy) \geq \mu(i) = 1$ implies $(x + i)y - xy \in I$, proving that I is a right ideal of R. The proof of converse is similar to that of Theorem 2.6. □

DEFINITION 2.10. Let R and S be near-rings. A map $\theta : R \to S$ is called a (*near-ring*) *homomorphism* if $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x)\theta(y)$ for any $x, y \in R$.

DEFINITION 2.11. If μ is a fuzzy set in R, and f is a function defined on R, then the fuzzy set ν in $f(R)$ defined by

$$\nu(y) = \sup_{x \in f^{-1}(y)} \mu(x)$$

for all $y \in f(R)$ is called the *image* of μ under f. Similarly, if ν is a fuzzy set in $f(R)$, then the fuzzy set $\mu = \nu \circ f$ in R (that is, the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in R$) is called the *preimage* of ν under f.

THEOREM 2.12. *A near-ring homomorphic preimage of a fuzzy left (right) ideal is a fuzzy left (right) ideal.*

Proof. Let $\theta : R \to S$ be a near-ring homomorphism, and ν be a fuzzy left ideal of S and μ the preimage of ν under θ. Then

$$\begin{aligned}\mu(x-y) &= \nu(\theta(x-y)) \\ &= \nu(\theta(x)-\theta(y)) \\ &\geq \min\{\nu(\theta(x)), \nu(\theta(y))\} \\ &= \min\{\mu(x), \mu(y)\},\end{aligned}$$

and

$$\begin{aligned}\mu(xy) &= \nu(\theta(xy)) \\ &= \nu(\theta(x)\theta(y)) \\ &\geq \nu(\theta(y)) \\ &= \mu(y),\end{aligned}$$

and

$$\begin{aligned}\mu(y+x-y) &= \nu(\theta(y+x-y)) \\ &= \nu(\theta(y)+\theta(x)-\theta(y)) \\ &\geq \nu(\theta(x)) \\ &= \mu(x)\end{aligned}$$

for all $x, y \in R$. Suppose that ν is a fuzzy right ideal of S. Then

$$\begin{aligned}\mu((x+i)y-xy) &= \nu(\theta((x+i)y-xy)) \\ &= \nu((\theta(x)+\theta(i))\theta(y)-\theta(x)\theta(y)) \\ &\geq \nu(\theta(i)) \\ &= \mu(i)\end{aligned}$$

for any $x, y, i \in R$. This proves the theorem. $\square$

We say that a fuzzy set μ in R has the *sup property* if, for any subset T of R, there exists $t_0 \in T$ such that

$$\mu(t_0) = \sup_{t \in T} \mu(t).$$

THEOREM 2.13. *A near-ring homomorphic image of a fuzzy left (right) ideal having the sup property is a fuzzy left (right) ideal.*

Proof. Let $\theta : R \to S$ be a near-ring homomorphism and μ be a fuzzy left ideal of R with the sup property and ν be the image of μ under θ. Given $\theta(x), \theta(y) \in \theta(R)$, let $x_0 \in \theta^{-1}(\theta(x))$, $y_0 \in \theta^{-1}(\theta(y))$ be such that

$$\mu(x_0) = \sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \qquad \mu(y_0) = \sup_{t \in \theta^{-1}(\theta(y))} \mu(t),$$

respectively. Then

$$\begin{aligned} \nu(\theta(x) - \theta(y)) &= \sup_{t \in \theta^{-1}(\theta(x) - \theta(y))} \mu(t) \\ &\geq \mu(x_0 - y_0) \\ &\geq \min\{\mu(x_0), \mu(y_0)\} \\ &= \min\{\sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \sup_{t \in \theta^{-1}(\theta(y))} \mu(t)\} \\ &= \min\{\nu(\theta(x)), \nu(\theta(y))\}, \end{aligned}$$

and

$$\begin{aligned} \nu(\theta(x)\theta(y)) &= \sup_{t \in \theta^{-1}(\theta(x)\theta(y))} \mu(t) \\ &\geq \mu(x_0 y_0) \\ &\geq \mu(y_0) \\ &= \sup_{t \in \theta^{-1}(\theta(y))} \mu(t) \\ &= \nu(\theta(y)), \end{aligned}$$

and

$$\begin{aligned} \nu(\theta(y + x - y)) &= \nu(\theta(y) + \theta(x) - \theta(y)) \\ &= \sup_{t \in \theta^{-1}(\theta(y) + \theta(x) - \theta(y))} \mu(t) \\ &\geq \mu(y_0 + x_0 - y_0) \\ &= \mu(x_0) \\ &= \sup_{t \in \theta^{-1}(\theta(x))} \mu(t) \\ &= \nu(\theta(x)). \end{aligned}$$

This proves that μ is a fuzzy left ideal of $\theta(R)$. Assume μ is a fuzzy right ideal of R. Given a $\theta(i) \in \theta(R)$, let $i_0 \in \theta^{-1}(\theta(i))$ such that $\mu(i_0) = \sup_{t\in\theta^{-1}(\theta(i))} \mu(t)$. Then

$$\begin{aligned}\nu(\theta((x+i)y - xy)) &= \nu((\theta(x)+\theta(i))\theta(y)) - \theta(x)\theta(y)) \\ &= \sup_{t\in\theta^{-1}((\theta(x)+\theta(i))\theta(y))-\theta(x)\theta(y))} \mu(t) \\ &\geq \mu((x_0+i_0)y_0 - x_0y_0) \\ &\geq \mu(i_0) \\ &= \sup_{t\in\theta^{-1}(\theta(i))} \mu(t) \\ &= \nu(\theta(i)),\end{aligned}$$

proves that ν is a fuzzy right ideal of $\theta(R)$. □

ACKNOWLEDGEMENTS. The authors express their thanks to the referee for his valuable help and concern.

References

1. S. Abou-Zaid, *On fuzzy subnear-rings and ideals*, Fuzzy Sets and Sys. **44** (1991), 139-146.
2. Yuan Bo and Wu Wangming, *Fuzzy ideals on a distributive lattice*, Fuzzy Sets and Sys. **35** (1990), 231-240.
3. J. R. Clay, *Nearrings; Geneses and Applications*, Oxford, New York (1992).
4. P. S. Das, *Fuzzy groups and level subgroups*, J. Math. Anal. and Appl. **84** (1981), 264-269.
5. V. N. Dixit, R. Kumar and N. Ajal, *On fuzzy rings*, Fuzzy Sets and Sys. **49** (1992), 205-213.
6. R. Kumar, *Fuzzy irreducible ideals in rings*, Fuzzy Sets and Sys. **42** (1991), 369-379.
7. R. Kumar, *Certain fuzzy ideals of rings redefined*, Fuzzy Sets and Sys. (1992), 251-260.
8. N. Kuroki, *Fuzzy bi-ideals in semigroups*, Comment. Math. Univ. St. Pauli. **28** (1979), 17-21.
9. N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Sys. **5** (1981), 203-215.
10. N. Kuroki, *Fuzzy semiprime ideals in semigroups*, Fuzzy Sets and Sys. **8** (1981), 71-79.

11. W. Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Sets and Sys. **8** (1982), 133-139.
12. D. S. Mailk, *Fuzzy ideals of artinian rings*, Fuzzy Sets and Sys. **37** (1990), 111-115.
13. R. G. McLean and H. Kummer, *Fuzzy ideals in semigroups*, Fuzzy Sets and Sys. **48** (1992), 137-140.
14. X. Ougen, *Fuzzy BCK-algebras*, Math. Japonica **36** (1991), 935-942.
15. A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512-517.
16. Z. Yue, *Prime L-fuzzy ideals and primary L-fuzzy ideals*, Fuzzy Sets and Sys. **27** (1988), 345-350.

SEUNG DONG KIM
DEPARTMENT OF MATHEMATICS EDUCATION, KONGJU NATIONAL UNIVERSITY, KONGJU 314-701, KOREA

HEE SIK KIM
DEPARTMENT OF MATHEMATICS EDUCATION, CHUNGBUK NATIONAL UNIVERSITY, CHONGJU 361-763, KOREA

E-mail: heekim@cbucc.chungbuk.ac.kr