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# A NEW EQUILIBRIUM EXISTENCE VIA CONNECTEDNESS

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In 1950, Nash [5] first proved the existence of equilibrium for games where the player's preferences are representable by continuous quasi-concave utilities and the strategy sets are simplexes. Next Debreu [3] proved the existence of equilibrium for abstract economies. Recently, the existence of Nash equilibrium can be further generalized in more general settings by several authors, e.g. Shafer-Sonnenschein [6], Borglin-Keiding [2], Yannelis-Prabhakar [8]. In the above results, the convexity assumption is very essential and the main proving tools are the continuous selection technique and the existence of maximal elements. Still there have been a number of generalizations and applications of equilibrium existence theorem in generalized games.

In this note, we first give a new maximal element existence theorem using the connectedness and next we shall prove a new equilibrium existence theorem for non-compact non-convex 1-person game. We also give an example that the previous results due to Shafer-Sonnenschein [6], Borglin-Keiding [2], Yannelis-Prabhakar [8], Tian [7] do not work; however our result can be applicable.

We first recall the following notations and definitions. Let $A$ be a non-empty set. We shall denote by $2^A$ the family of all subsets of $A$. Let $X, Y$ be non-empty topological spaces and $T : X \to 2^Y$ be a correspondence. Then $T$ is said to be *open* or have *open graph* (respectively, *closed* or *closed graph*) if the graph of $T$ ( $\mathrm{Gr}\, T = \{(x,y) \in X \times Y \mid y \in T(x)\}$) is open (respectively, closed) in $X \times Y$. We may call $T(x)$ the

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*upper section* of $T$, and $T^{-1}(y)(= \{x \in X \mid y \in T(x)\})$ the *lower section* of $T$. It is easy to check that if $T$ has open graph, then the upper and lower sections of $T$ are open ; however the converse is not true in general. A multimap $T : X \to 2^Y$ is said to be *closed at* $x$ if for each net $(x_\alpha) \to x$, $y_\alpha \in T(x_\alpha)$ and $(y_\alpha) \to y$, then $y \in T(x)$. And $T$ is *closed* on $X$ if it is closed at every point of $X$. Note that if $T$ is single-valued, then the closedness is equivalent to continuity as a function. A correspondence $T : X \to 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set $V$ in $Y$ with $T(x) \subset V$, then there exists an open neighborhood $U$ of $x$ in $X$ such that $T(y) \subset V$ for each $y \in U$. It is easy to see that when $X$ and $Y$ are regular topological spaces and $T$ is upper semicontinuous and each $T(x)$ is non-empty closed, then $T$ has closed graph; so $T$ is closed (for the proof, see Proposition 11.9 of Border [1]).

Let $T : X \to 2^Y$ be a correspondence; then $x \in X$ is called a *maximal element* for $T$ if $T(x) = \emptyset$. Indeed, in real applications, the maximal element may be interpreted as the set of those objects in $X$ that are the "best" or "largest" choices.

Let $I$ be a (possibly uncountable) set of agents. For each $i \in I$, let $X_i$ be a non-empty set of actions. A *generalized game* (or *an abstract economy*) $\Gamma = (X_i, A_i, P_i)_{i \in I}$ is defined as a family of ordered triples $(X_i, A_i, P_i)$ where $X_i$ is a non-empty topological space (a choice set), $A_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a constraint correspondence and $P_i : \Pi_{j \in I} X_j \to 2^{X_i}$ is a preference correspondence. An *equilibrium* for $\Gamma$ is a point $\hat{x} \in X = \Pi_{i \in I} X_i$ such that for each $i \in I$, $\hat{x}_i \in A_i(\hat{x})$ and $P_i(\hat{x}) \cap A_i(\hat{x}) = \emptyset$. In particular, when $I = \{1, \cdots, n\}$, we may call $\Gamma$ an *N-person game.*

We begin with the following :

LEMMA. *Let $X$ be a non-empty connected subset of a Hausdorff topological space $E$ and $T : X \to 2^X$ be closed at every $x$, where $T(x) \neq \emptyset$, such that*

(1) *$T^{-1}(y_o)$ is non-empty open in $X$ for some $y_o \in X$,*

(2) *$x \notin T(x)$ for each $x \in X$.*

*Then $T$ has a maximal element $\hat{x} \in X$, i.e., $T(\hat{x}) = \emptyset$.*

*Proof.* Suppose the assertion were false. Then $T(x)$ is non-empty for each $x \in X$ and so $T$ is closed at every $x \in X$. Since $T$ is closed, the lower section $T^{-1}(y_o)$ is closed. In fact, for every net $(x_\alpha)_{\alpha \in \Gamma} \subset T^{-1}(y_o)$ with $(x_\alpha) \to x$, we have $y_o \in T(x_\alpha)$ for each $\alpha \in \Gamma$ and $(x_\alpha) \to x$, so by the closedness of $T$ at $x$, $y_o \in T(x)$. Hence $x \in T^{-1}(y_o)$, so $T^{-1}(y_o)$ is closed. By the assumption (1), $T^{-1}(y_o)$ is also non-empty open. Therefore, by the connectedness of $X$, $T^{-1}(y_o) = X$. Hence we have $y_o \in T(x)$ for each $x \in X$ and hence $y_o \in T(y_o)$, which contradicts the assumption (2). Therefore $T$ has a maximal element $\hat{x} \in X$, i.e. $T(\hat{x}) = \emptyset$. This completes the proof.

It should be noted that in the above Lemma, we do not need the compact convex assumption on $X$ and also do not need the closed convex assumption on $T(x)$; but we shall need the non-empty open lower section at some special point.

The following simple example is suitable for our Lemma:

EXAMPLE 1. Let $X = \{(x, y) \in R^2 \mid 0 \le x,\ 0 \le y \le \frac{1}{x}\}$ be a connected set in $R^2$ and a correspondence $T : X \to 2^X$ be defined as follows :

$$T(x,y) := \begin{cases} \text{line segment from } (0{,}0) \text{ to } \dfrac{1}{2}(x,y), & \text{if } (x,y) \neq (0,0), \\ \emptyset, & \text{if } (x,y) = (0,0). \end{cases}$$

Then it is easy to show that the correspondence $T$ is closed at every $(x, y) \neq (0,0)$ and $(x, y) \notin T(x, y)$ for each $(x, y) \in X$. And note that $T^{-1}(0,0) = X \setminus (0,0)$ is open in $X$. Therefore, by Lemma, $T$ has a maximal element $(0,0)$ in $X$.

Using Lemma, we shall prove a basic new equilibrium existence theorem for a connected 1-person game.

THEOREM. *Let $\Gamma = (X, A, P)$ be an 1-person game such that*

*(1) $X$ is a non-empty connected subset of a regular topological space,*

*(2) the correspondence $A : X \to 2^X$ is upper semicontinuous such that for each $x \in X$, $A(x)$ is non-empty closed in $X$,*

*(3) the correspondence $P : X \to 2^X$ is upper semicontinuous such that $P(x)$ is closed in $X$ for each $x \in X$, and $P(x)$ is non-empty for each $x \notin \mathcal{F} := \{x \in X : x \in A(x)\}$,*

(4) *for some* $y_o \in X$, $A^{-1}(y_o)$ *and* $A^{-1}(y_o) \cap P^{-1}(y_o)$ *are non-empty open in* $X$,

(5) *for each* $x \in X$, $x \notin P(x)$.

*Then* $\Gamma$ *has an equilibrium choice* $\hat{x} \in X$, *i.e.*,
$\hat{x} \in A(\hat{x})$ *and* $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

*Proof.* Note that since $A$ is closed and the assumptions (2) and (4), by using Lemma, the fixed point set $\mathcal{F}$ of $A$ is non-empty closed.

We now define a correspondence $\phi : X \to 2^X$ by

$$\phi(x) = \begin{cases} P(x), & \text{if} \quad x \notin \mathcal{F}, \\ A(x) \cap P(x), & \text{if} \quad x \in \mathcal{F}. \end{cases}$$

Then, by the assumption (5), we have $x \notin \phi(x)$ for each $x \in X$. We shall show that $\phi$ is upper semicontinuous. Let $V$ be any open subset of $X$ containing $\phi(x)$. Then we let

$$\begin{aligned} U &:= \{x \in X : \phi(x) \subset V\} \\ &= \{x \in \mathcal{F} : \phi(x) \subset V\} \cup \{x \in X \setminus \mathcal{F} : \phi(x) \subset V\} \\ &= \{x \in \mathcal{F} : (A \cap P)(x) \subset V\} \cup \{x \in X \setminus \mathcal{F} : P(x) \subset V\} \\ &= \{x \in X : (A \cap P)(x) \subset V\} \cup \{x \in X \setminus \mathcal{F} : P(x) \subset V\}. \end{aligned}$$

Since $X \setminus \mathcal{F}$ is open, $P$ is upper semicontinuous and $A \cap P$ is upper semicontinuous at every $x$ with $(A \cap P)(x) \neq \emptyset$, $U$ is open and hence $\phi$ is also upper semicontinuous at every $x$ with $\phi(x) \neq \emptyset$. Since each $\phi(x)$ is closed, by Proposition 11.9 of Border [1], $\phi$ is closed at every $x \in X$ with $\phi(x) \neq \emptyset$.

Next we shall show that $\phi^{-1}(y_o)$ is an open subset of $X$. In fact, by the assumption (4), we have that

$$\begin{aligned} \phi^{-1}(y_o) &= \{x \in X : y_o \in \phi(x)\} \\ &= \{x \in \mathcal{F} : y_o \in \phi(x)\} \cup \{x \in X \setminus \mathcal{F} : y \in \phi(x)\} \\ &= [\mathcal{F} \cap (A \cap P)^{-1}(y_o)] \cup [(X \setminus \mathcal{F}) \cap P^{-1}(y_o)] \\ &= P^{-1}(y_o) \cap [A^{-1}(y_o) \cup ((X \setminus \mathcal{F}) \cap P^{-1}(y_o))] \\ &= [P^{-1}(y_o) \cap A^{-1}(y_o)] \cup [(X \setminus \mathcal{F}) \cap P^{-1}(y_o)] \end{aligned}$$

is non-empty open in $X$. Therefore, by Lemma, there exists a point $\hat{x} \in X$ such that $\phi(\hat{x}) = \emptyset$. If $\hat{x} \notin \mathcal{F}$, then $\phi(\hat{x}) = P(\hat{x}) = \emptyset$, which is a contradiction. Therefore, we have $\hat{x} \in \mathcal{F}$ and $\phi(\hat{x}) = A(\hat{x}) \cap P(\hat{x}) = \emptyset$, i.e., $\hat{x} \in A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. This completes the proof.

REMARK. Our Theorem is quite different from the previous many equilibrium existence theorems (e.g. Shafer-Sonnenschein [6], Borglin-Keiding [2], Yannelis-Prabhakar [8], Kim [4]). In these results, the compactness and convexity assumptions are very essential. But we do not need any compact convex assumption on the choice set $X$, but we only need the connectedness assumption. Also we do not need the convexity assumptions on the values $A(x)$ and $P(x)$ and strong open lower section assumptions; but we need the weaker open lower section property at some special point.

Next we give an example of a connected 1-person game where our Theorem can be applicable but the previous known results can not be applicable:

EXAMPLE 2. Let $X = \{(x,y) \in R^2 \mid 0 \leq x, 0 \leq y \leq \frac{1}{x}\}$ be a connected choice set and the correspondences $A, P : X \to 2^X$ be defined as follows:

$$A(x,y) := \{(s,t) \mid s = y, 0 \leq t \leq \frac{1}{y} \text{ or } 0 \leq s \leq y, t = 0\},$$

for each $(x,y) \in X$,

$$P(x,y) := \begin{cases} \emptyset, & \text{for each } (x,x) \in X \text{ with } 0 \leq x \leq 1, \\ \{(s,t) \mid s = y, 0 \leq t \leq \frac{1}{y} \text{ or } 0 \leq s \leq y, t = 0\}, & \text{otherwise.} \end{cases}$$

Here, we shall use $1/0$ as the infinity for simplicity of the formula. Then it is easy to show that the correspondence $A$ is upper semicontinuous and each $A(x,y)$ is non-empty closed and the fixed point set $\mathcal{F}$ of $A$ is exactly the diagonals of $X$, i.e., $\mathcal{F} = \{(x,x) \mid 0 \leq x \leq 1\}$. Also we have that $P$ is upper semicontinuous on $X \setminus \mathcal{F}$ and $P(x,y)$ is non-empty closed at every point except on the diagonals. And note that

$A^{-1}(0,0) = X$ is open and $P^{-1}(0,0) = X \setminus \mathcal{F}$ is also open. Therefore all assumptions of Theorem are satisfied, so that we can obtain an equilibrium point $(0,0) \in X$ such that $(0,0) \in A(0,0)$ and $A(0,0) \cap P(0,0) = \emptyset$.

Finally, it should be noted that by modifying the methods in Borglin-Keiding [2] or Kim [4], we can show that the case of N-agents can be reduced to the 1-person game.

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