Bull. Korean Math. Soc. 33 (1996), No. 4, pp. 565–585

SOME MINIMIZATION THEOREMS IN GENERATING SPACES OF QUASI-METRIC FAMILY AND APPLICATIONS

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1. Introduction

In 1976, Caristi [1] established a celebrated fixed point theorem in complete metric spaces, which is a very useful tool in the theory of nonlinear analysis. Since then, several generalizations of the theorem were given by a number of authors: for instances, generalizations for single-valued mappings were given by Downing and Kirk [4], Park [11] and Siegel [13], and the multi-valued versions of the theorem were obtained by Chang and Luo [3], and Mizoguchi and Takahashi [10].

Later, Takahashi [14] proved a nonconvex minimization theorem in the complete metric spaces, which was used to obtain Caristi's fixed point theorem and some existence theorems. In particular, Ume [15] generalized the results of Takahashi [14] using a continuous mapping satisfying a certain condition.

On the other hand, Kaleva and Seikkala [9] introduced a concept of a fuzzy metric space which generalizes the notion of a metric space by setting the distance between two points to be a nonnegative fuzzy number, and proved some fixed point theorems. Since then, Jung et al. [8] established a Takahashi-type minimization theorem [14] in complete fuzzy metric spaces. By using their minimization theorem, they obtained the analogoue of Downing-Kirk's fixed point theorem.

Received April 8, 1996.

1991 AMS Subject Classification: 47H10, 54C60, 58E30, 54A40, 54E70.

Key words and phrases: Fixed point theorem, fuzzy metric space, generating spaces of quasi-metric family, minimization theorems, probabilistic metric space.

This paper was supported by NON-DIRECTED RESEARCH FUND, Korea Research Foundation, 1995-1996.

Very recently, Chang et al. [2] gave a defintion of generating spaces of quasi-metric family, which generalizes those of fuzzy metric spaces in the sense of Kaleva and Seikkala [9] and Menger probabilstic metric spaces [12], and some properties and examples of the spaces. They also gave several fixed point theorems and Takahashi-type minimization theorems in complete generating spaces of quasi-metric family.

In this paper, we establish new nonconvex minimization theorems in complete genereating spaces of quasi-metric family. As consequences, we generealize results in [14, 15] and Downing-Kirk's fixed point theorem in the same spaces. Simultaneously, applying these results to fuzzy metric spaces and probabilistic metric spaces, we give the corresponding results. Our results extend and improve upon the corresponding results of [1, 4, 6, 7, 8, 14, 15].

2. Minimization Theorems in Generating Spaces of Quasi-Metric Family

In this section, we first give new nonconvex minimization theorem in generating spaces of quasi-metric family. Then we show that our results improve those of [14, 15] and Downing-Kirk's fixed point theorem in the same spaces.

Now, we give the definition, some properties and examples of generating spaces of quasi-metric family.

DEFINITON 2.1. [2] Let X be a nonempty set and $\{d_\alpha : \alpha \in (0,1]\}$ be a family of mappings d_α of $X \times X$ into $\mathbb{R}^+$. $(X, d_\alpha : \alpha \in (0,1])$ is called a *generating space of quasi-metric family* if it satisfies the following conditions:

(QM-1) $d_\alpha(x,y) = 0$ for all $\alpha \in (0,1]$ if and only if $x = y$,

(QM-2) $d_\alpha(x,y) = d_\alpha(y,x)$ for all $x,\ y \in X$ and $\alpha \in (0,1]$,

(QM-3) For any $\alpha \in (0,1]$, there exists a number $\mu \in (0,\alpha]$ such that

$$d_\alpha(x,y) \le d_\mu(x,z) + d_\mu(z,y), \quad x,\ y,\ z \in X,$$

(QM-4) For any $x,\ y \in X$, $d_\alpha(x,y)$ is nonincreasing and left continuous in α.

In what follows $\{d_\alpha : \alpha \in (0,1]\}$ will be called a family of quasi-metrics.

EXAMPLE 2.1. Let (X, d) be a metric space. Letting $d_\alpha(x, y) = d(x, y)$ for all $\alpha \in (0, 1]$ and $x, y \in X$, then (X, d) is a generating space of quasi-metric family. Furthermore, both every fuzzy metric space (see Definition 3.1) and every probabilistic metric space (see Definition 4.1) are the examples of generating spaces of quasi-metric family (its proof will be given in the sections 3 and 4 below).

In [5], Fan proved that if $(X, d_\alpha : \alpha \in (0, 1])$ is a generating space of quasi-metric family, then there exists a topology $\mathcal{T}_{\{d_\alpha\}}$ on X such that $(X, \mathcal{T}_{\{d_\alpha\}})$ is a Hausdorff topological space and $\mathcal{U}(x) = \{U_x(\epsilon, \alpha) : \epsilon > 0,\ \alpha \in (0, 1]\}$, $x \in X$, is a basis of neighborhoods of the point x for the topology $\mathcal{T}_{\{d_\alpha\}}$, where

$$U_x(\epsilon, \alpha) = \{y \in X : d_\alpha(x, y) < \epsilon\}.$$

Throughout the following, we assume that $k : (0, 1] \to (0, \infty)$ is a nondecreasing function satisfying the following condition:

$$M = \sup_{\alpha \in (0,1]} k(\alpha) < \infty. \tag{2.1}$$

Now, we give our main results.

THEOREM 2.1. *Let $(X, d_\alpha : \alpha \in (0, 1])$ and $(Y, \delta_\alpha : \alpha \in (0, 1])$ be two complete generating spaces of quasi-metric family, $f : X \to Y$ a closed mapping, and $T : X \to X$ a continuous mapping satisfying $d_\alpha(Tx, Ty) \le d_\alpha(x, Ty)$ and $\delta_\alpha(f(Tx), f(Ty)) \le \delta_\alpha(f(x), f(Ty))$ for every $x,\ y \in X$ and $\alpha \in (0, 1]$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function, bounded from below and $\phi : f(X) \to \mathbb{R}$ a lower semicontinuous function, bounded from below. Assume that for any $u \in X$ with $\inf_{x \in X} \psi(\phi(f(x))) < \psi(\phi(f(u)))$, there exists $v \in X$ with $v \ne Tu$ and*

$$\max\{d_\alpha(v, Tu) + d_\alpha(Tu, Tv), c[\delta_\alpha(f(v), f(Tu)) + \delta_\alpha(f(Tu), f(Tv))]\} \\ \le k(\alpha)[\psi(\phi(f(u))) - \psi(\phi(f(v)))]$$

for any $\alpha \in (0, 1]$, where c is a given constant. Then there exists an $x_0 \in X$ such that

$$\inf_{x \in X} \psi(\phi(f(x))) = \psi(\phi(f(x_0))).$$

Proof. Suppose $\inf_{x\in X}\psi(\phi(f(x))) < \psi(\phi(f(y)))$ for every $y \in X$ and choose $u \in X$ with $\psi(\phi(f(u))) < \infty$. Then we define inductively a sequence $\{u_n\} \subset X$ with $u_1 = u$. Suppose that $u_n \in X$ is known. Then choose $u_{n+1} \in W_n$ such that

$$
\begin{aligned}
W_n = \{w \in X : \max\{&d_\alpha(w, Tu_n) + d_\alpha(Tu_n, Tw), \\
&c[\delta_\alpha(f(w), f(Tu_n)) + \delta_\alpha(f(Tu_n), f(Tw))]\} \\
&\leq k(\alpha)[\psi(\phi(f(u_n))) - \psi(\phi(f(w)))]\}
\end{aligned}
\tag{2.2}
$$

for any $\alpha \in (0,1]$ and

$$
\begin{aligned}
\psi(\phi(f(u_{n+1}))) \leq & \inf_{x\in W_n}\psi(\phi(f(x))) \\
&+ \frac{1}{2}[\psi(\phi(f(u_n))) - \inf_{x\in W_n}\psi(\phi(f(x)))].
\end{aligned}
\tag{2.3}
$$

Now we prove that $\{u_n\}$ and $\{f(u_n)\}$ are Cauchy sequences in X and Y, respectively. In fact, if $n < m$, by the construction of the sequence $\{u_n\}$ and the hypothesis, we obtain

$$
\begin{aligned}
&\max\{2d_\alpha(Tu_n, Tu_m), 2c\delta_\alpha(f(Tu_n), f(Tu_m))\} \\
&\leq \sum_{j=n}^{m-1}\max\{2d_{\alpha_j}(Tu_j, Tu_{j+1}), 2c\delta_{\alpha_j}(f(Tu_j), f(Tu_{j+1}))\} \\
&\leq \sum_{j=n}^{m-1}\max\{d_{\alpha_j}(u_{j+1}, Tu_j) + d_{\alpha_j}(Tu_j, Tu_{j+1}), \\
&\qquad c[\delta_{\alpha_j}(f(u_{j+1}), f(Tu_j)) + \delta_{\alpha_j}(f(Tu_j), f(Tu_{j+1}))]\} \\
&\leq \sum_{j=n}^{m-1} k(\alpha_j)[\psi(\phi(f(u_j))) - \psi(\phi(f(u_{j+1})))] \\
&\leq k(\alpha)[\psi(\phi(f(u_n))) - \psi(\phi(f(u_m)))]
\end{aligned}
\tag{2.4}
$$

for some α_j with $0 < \alpha_{j+1} \leq \alpha_j \leq \alpha$, $j = n, \cdots, m-1$, and from (2.4), we obtain

$$\begin{aligned}
&\max\{d_\alpha(u_n,Tu_m),c\delta_\alpha(f(u_n),f(Tu_m))\}\\
\le\ &\max\{d_\mu(u_n,Tu_n)+2d_\mu(Tu_n,Tu_m)\\
&\qquad c[\delta_\mu(f(u_n),f(Tu_n))+2\delta_\mu(f(Tu_n),f(Tu_m))]\}\\
\le\ &\max\{d_\mu(u_n,Tu_n),c\delta_\mu(f(u_n),f(Tu_n))\}\\
&\qquad+\max\{2d_\mu(Tu_n,Tu_m),2c\delta_\mu(f(Tu_n),f(Tu_m))\}\\
\le\ &\max\{d_\mu(u_n,Tu_n)+d_\mu(Tu_n,Tu_n),\\
&\qquad c[\delta_\mu(f(u_n),f(Tu_n))+\delta_\mu(f(Tu_n),f(Tu_n))]\}\\
&\qquad+k(\mu)[\psi(\phi(f(u_n)))-\psi(\phi(f(u_m)))]\\
\le\ &k(\mu)[\psi(\phi(f(u_n)))-\psi(\phi(f(u_n)))\\
&\qquad+\psi(\phi(f(u_n)))-\psi(\phi(f(u_m)))]\\
\le\ &k(\alpha)[\psi(\phi(f(u_n)))-\psi(\phi(f(u_m)))]
\end{aligned}\tag{2.5}$$

for some $\mu\in(0,\alpha]$. Since $\{\psi(\phi(f(u_n)))\}$ is monotonically decreasing by (2.2), we also have

$$\begin{aligned}
&\max\{d_\alpha(u_m,Tu_m),c\delta_\alpha(f(u_m),f(Tu_m))\}\\
=\ &\max\{d_\alpha(u_m,Tu_m)+d_\alpha(Tu_m,Tu_m),\\
&\qquad c[\delta_\alpha(f(u_m),f(Tu_m))+\delta_\alpha(f(Tu_m),f(Tu_m))]\}\\
\le\ &k(\alpha)[\psi(\phi(f(u_m)))-\psi(\phi(f(u_m)))]\\
=\ &0\\
\le\ &k(\alpha)[\psi(\phi(f(u_n)))-\psi(\phi(f(u_m)))]
\end{aligned}\tag{2.6}$$

for any $\alpha\in(0,1]$. Hence, from (2.5) and (2.6), we obtain

$$\begin{aligned}
&\max\{d_\alpha(u_n,u_m),c\delta_\alpha(f(u_n),f(u_m))\}\\
\le\ &\max\{d_\mu(u_n,Tu_m)+d_\mu(Tu_m,u_m),\\
&\qquad c[\delta_\mu(f(u_n),f(Tu_m))+\delta_\mu(f(Tu_m),f(u_m))]\}\\
\le\ &\max\{d_\mu(u_n,Tu_m),c\delta_\mu(f(u_n),f(Tu_m))\}\\
&\qquad+\max\{d_\mu(u_m,Tu_m),c\delta_\mu(f(u_m),f(Tu_m))\}\\
\le\ &2k(\mu)[\psi(\phi(f(u_n)))-\psi(\phi(f(u_m)))]\\
\le\ &2k(\alpha)[\psi(\phi(f(u_n)))-\psi(\phi(f(u_m)))]
\end{aligned}$$

for any $\alpha \in (0,1]$ and some $\mu \in (0,\alpha]$. On the other hand, by the boundedness of ψ from below , there exists a finite number γ such that $\psi(\phi(f(u_n))) \downarrow \gamma$. Also from the nondecreasing and the continuity of ψ, it follows that $\lim_{n\to\infty} \phi(f(u_n))$ exists. Hence for any given $\lambda > 0$ and $\varepsilon > M\lambda$ (where M is a constant defined by (2.1)). there exists an n_0 such that for $n \geq n_0$ we have

$$\gamma \leq \psi(\phi(f(u_n))) < \gamma + \frac{\lambda}{2}.$$

Thus for any $m > n \geq n_0$, we have

$$0 \leq \psi(\phi(f(u_n))) - \psi(\phi(f(u_m))) < \gamma + \frac{\lambda}{2} - \gamma = \frac{\lambda}{2}.$$

Thus, for any $\alpha \in (0,1]$ and $m > n \geq n_0$, we have

$$\begin{aligned} &\max\{d_\alpha(u_n, u_m), c\delta_\alpha(f(u_n), f(u_m))\} \\ &\quad\leq\ 2k(\alpha)[\psi(\phi(f(u_n))) - \psi(\phi(f(u_m)))] \\ &\quad<\ M\lambda < \varepsilon. \end{aligned}$$

This implies that $\{u_n\}$ and $\{f(u_n)\}$ are Cauchy sequences in X and Y, respectively. By the completeness of X and Y, there exist $u \in X$ and $\bar{v} \in Y$ such that $\lim_{n\to\infty} u_n = \bar{u}$ and $\lim_{n\to\infty} f(u_n) = \bar{v}$. Since f is a closed mapping, $f(\bar{u}) = \bar{v} \in f(X)$. By the lower semicontinuity and boundedness of ϕ from below, we have

$$-\infty < \phi(f(\bar{u})) \leq \liminf_{n\to\infty} \phi(f(u_n)) = \lim_{n\to\infty} \phi(f(u_n)).$$

So, from the nondecreasingness and the continuity of ψ, it follows that

$$\psi(\phi(f(\bar{u}))) \leq \lim_{n\to\infty} \psi(\phi(f(u_n))) = \gamma \leq \psi(\phi(f(u_n))) \tag{2.7}$$

for any n. Letting $m \to \infty$ in (2.4) and (2.6), we obtain for any $\alpha \in (0,1]$

$$\begin{aligned} &\max\{2d_\alpha(Tu_n, T\bar{u}), 2c\delta_\alpha(f(Tu_n), f(T\bar{u}))\} \\ &\quad\leq k(\alpha)[\psi(\phi(f(u_n))) - \psi(\phi(f(\bar{u})))], \end{aligned}$$

$u = T\bar{u}$ and $f(\bar{u}) = f(T\bar{u})$, respectively. On the other hand, by hypothesis, there exists a $v \in X$ such that $v \neq T\bar{u}$ and

$$\begin{aligned} \max\{&d_\alpha(v, T\bar{u}) + d_\alpha(T\bar{u}, Tv), \\ &c[\delta_\alpha(f(v), f(T\bar{u})) + \delta_\alpha(f(T\bar{u}), f(Tv))]\} \\ &\leq k(\alpha)[\psi(\phi(f(\bar{u}))) - \psi(\phi(f(v)))] \end{aligned} \tag{2.8}$$

for any $\alpha \in (0, 1]$. Hence we have

$$\begin{aligned} \max\{&d_\alpha(v, Tu_n) + d_\alpha(Tu_n, Tv), \\ &c[\delta_\alpha(f(v), f(Tu_n)) + \delta_\alpha(f(Tu_n), f(Tv))]\} \\ \leq \ &\max\{d_\mu(v, T\bar{u}) + d_\mu(T\bar{u}, Tv) + 2d_\mu(Tu_n, T\bar{u}), \\ &\quad c[\delta_\mu(f(v), f(T\bar{u})) + \delta_\mu(f(T\bar{u}), f(Tv)) + 2\delta_\mu(f(Tu_n), f(T\bar{u}))]\} \\ \leq \ &\max\{d_\alpha(v, T\bar{u}) + d_\alpha(T\bar{u}, Tv), \\ &\quad c[\delta_\alpha(f(v), f(T\bar{u})) + \delta_\alpha(f(T\bar{u}), f(Tv))]\} \\ &\quad + \max\{2d_\mu(Tu_n, T\bar{u}), 2\delta_\mu(f(Tu_n), f(T\bar{u}))\} \\ \leq \ &k(\mu)[\psi(\phi(f(\bar{u}))) - \psi(\phi(f(v))) + \psi(\phi(f(u_n))) - \psi(\phi(f(\bar{u})))] \\ \leq \ &k(\alpha)[\psi(\phi(f(u_n))) - \psi(\phi(f(v)))] \end{aligned}$$

for any $\alpha \in (0, 1]$ and some $\mu \in (0, \alpha]$. This implies $v \in W_n$. Using (2.3), we have

$$2\psi(\phi(f(u_{n+1}))) - \psi(\phi(f(u_n))) \leq \inf_{x \in W_n} \psi(\phi(f(x))) \leq \psi(\phi(f(v))).$$

Thus, by (2.7) and (2.8), we obtain

$$\psi(\phi(f(v))) < \psi(\phi(f(\bar{u}))) \leq \lim_{n \to \infty} \psi(\phi(f(u_n))) \leq \psi(\phi(f(v))).$$

This is a contradiction. Therefore, there exists an $x_0 \in X$ such that

$$\psi(\phi(f(x_0))) = \inf_{x \in X} \psi(\phi(f(x))).$$

This completes the proof.

From Theorem 2.1, we can obtain the following Theorem 2.2 and Theorem 2.3 which generalizes the results of Chang et al. [2] and Ume [15], respectively.

THEOREM 2.2. *Let* $(X, d_\alpha : \alpha \in (0,1])$ *and* $(Y, \delta_\alpha : \alpha \in (0,1])$ *be two complete generating spaces of quasi-metric family,* $f : X \to Y$ *a closed mapping,* $\psi : \mathbb{R} \to \mathbb{R}$ *a nondecreasing continuous function, bounded from below and* $\phi : f(X) \to \mathbb{R}$ *a lower semicontinuous function, bounded from below. Assume that for any* $u \in X$ *with* $\inf_{x \in X} \psi(\phi(f(x))) < \psi(\phi(f(u)))$, *there exists* $v \in X$ *with* $v \neq u$ *and*

$$\max\{d_\alpha(v,u), c\delta_\alpha(f(v), f(u))\} \leq k(\alpha)[\psi(\phi(f(u))) - \psi(\phi(f(v)))]$$

for any $\alpha \in (0,1]$, *where* c *is a given constant. Then there exists an* $x_0 \in X$ *such that*

$$\inf_{x \in X} \psi(\phi(f(x))) = \psi(\phi(f(x_0))).$$

Proof. The result follows from Theorem 2.1 with $T = I$, where I denotes the identity operator on X.

THEOREM 2.3. *Let* $(X, d_\alpha : \alpha \in (0,1])$ *be a complete generating space of quasi-metric family,* $T : X \to X$ *a continuous mapping satisfying* $d_\alpha(Tx, Ty) \leq d_\alpha(x, Ty)$ *for every* $x,\ y \in X$ *and* $\alpha \in (0,1]$, *and* $\phi : X \to \mathbb{R}$ *a lower semicontinuous function, bounded from below. Assume that for any* $u \in X$ *with* $\inf_{x \in X} \phi(x) < \phi(u)$, *there exists* $v \in X$ *with* $v \neq Tu$ *and*

$$d_\alpha(v, Tu) + d_\alpha(Tu, Tv) \leq k(\alpha)[\phi(u) - \phi(v)]$$

for any $\alpha \in (0,1]$. *Then there exists an* $x_0 \in X$ *such that*

$$\inf_{x \in X} \phi(x) = \phi(x_0).$$

Proof. The result follows from Theorem 2.1 with $X = Y$, $f = I$, $\psi = I$ and $c = 1$.

REMARK 2.1. (1) Theorem 5.1 in [2] is a special case of Theorem 2.2 with $\psi = I$.

(2) Theorem 1 in [15] is a special case of Theorem 2.3 with X being a metric space and $k(\alpha) \equiv k$, where k is a given constant.

For a mapping $T : X \to X$, we denote by $F(T)$ the set of all fixed points of T and by $T(X)$ the range of the mapping T, respectively. By applying Theorem 2.1, we have the following theorem.

THEOREM 2.4. *Let* $(X, d_\alpha : \alpha \in (0,1])$ *and* $(Y, \delta_\alpha : \alpha \in (0,1])$ *be two complete generating spaces of quasi-metric family,* $f : X \to Y$ *a closed mapping and* $T : X \to X$ *a continuous mapping satisfying* $d_\alpha(Tx, Ty) \leq d_\alpha(x, Ty)$ *and* $\delta_\alpha(f(Tx), f(Ty)) \leq \delta_\alpha(f(x), f(Ty))$ *for every* x, $y \in X$ *and* $\alpha \in (0,1]$. *Let* $\psi : \mathbb{R} \to \mathbb{R}$ *be a nondecreasing continuous function, bounded from below and* $\phi : f(X) \to \mathbb{R}$ *a lower semicontinuous function, bounded from below. Let* $S : X \to X$ *be a mapping satisfying* $ST = TS$ *and*

$$\begin{aligned}\max\{&d_\alpha(Sx, Tx) + d_\alpha(Tx, TSx),\\ &c[\delta_\alpha(f(Sx), f(Tx)) + \delta_\alpha(f(Tx), f(TSx))]\}\\ &\leq k(\alpha)[\psi(\phi(f(x))) - \psi(\phi(f(Sx)))]\end{aligned}$$

for any $x \in X$ *and* $\alpha \in (0,1]$, *where* c *is a given constant. Then there exists an* $x_0 \in X$ *with* $Tx_0 = x_0 = Sx_0$.

Proof. From the hypothesis on the mapping T, it is easily shown that $T(X) = F(T)$. Suppose now that $Tx \neq Sx$ for every $x \in E$. Then, for every $x \in X$, there exists $Sx \in X$ such that $Tx \neq Sx$ and

$$\begin{aligned}\max\{&d_\alpha(Sx, Tx) + d_\alpha(Tx, TSx),\\ &c[\delta_\alpha(f(Sx), f(Tx)) + \delta_\alpha(f(Tx), f(TSx))]\}\\ &\leq k(\alpha)[\psi(\phi(f(x))) - \psi(\phi(f(Sx)))]\end{aligned}$$

For any $\alpha \in (0,1]$. So, from Theorem 2.1, we obtain a $z \in X$ with $\psi(\phi(f(z))) = \inf_{x\in X} \psi(\phi(f(x)))$. For such a $z \in X$, we have

$$\begin{aligned}0 &< d_\alpha(Sz, Tz) + d_\alpha(Tz, TSz)\\ &\leq \max\{d_\alpha(Sz, Tz) + d_\alpha(Tz, TSz),\\ &\qquad c[\delta_\alpha(f(Sz), f(Tz)) + \delta_\alpha(f(Tz), f(TSz))]\}\\ &\leq k(\alpha)[\psi(\phi(f(z))) - \psi(\phi(f(Sz)))]\\ &\leq 0\end{aligned}$$

for any $\alpha \in (0,1]$. This is a contradiction. Therefore there exists an $x_0 \in X$ with $Sx_0 = Tx_0$. Since $T(X) = F(T)$, we have $Sx_0 = TSx_0 = STx_0$ and hence $x_0 = Sx_0 = Tx_0$. This completes the proof.

As a direct consequence of Theorem 2.4, we obtain the following result, which is a generalized form of Downing-Kirk's fixed point theorem in generating spaces of quasi-metric family.

COROLLARY 2.5. *Let $(X, d_\alpha : \alpha \in (0,1])$ and $(Y, \delta_\alpha : \alpha \in (0,1])$ be two complete generating spaces of quasi-metric family, $f : X \to Y$ a closed mapping and $\phi : f(X) \to \mathbb{R}$ a lower semicontinuous function, bounded from below. Let $S : X \to X$ be a mapping satisfying*

$$\max\{d_\alpha(Sx, x), c\delta_\alpha(f(Sx), f(x))\} \leq k(\alpha)[\phi(f(x)) - \phi(f(Sx))]$$

for any $x \in X$ and $\alpha \in (0,1]$, where c is a given constant. Then there exists an $x_0 \in X$ with $Sx_0 = x_0$.

Proof. The result follows from Theorem 2.4 with $T = I$ and $\psi = I$.

REMARK 2.2. (1) When X, Y are metric spaces and $k(\alpha) \equiv 1$, from Corollary 2.5, we obtain Downing-Kirk's fixed point theorem [4]. Caristi's fixed point theorem [1] is also obtained from Corollary 2.5 with $X = Y$ being a metric space, $f = I$, $c = 1$ and $k(\alpha) \equiv 1$.

(2) When X is a metric space, an example of satisfying $d(Tx, Ty) \leq d(x, Ty)$ for every x, $y \in X$ is given in [15].

3. Versions in Fuzzy Metric Spaces

A mapping $x : \mathbb{R} \to [0,1]$ is called a *fuzzy number*. For $\alpha \in (0,1]$ and a fuzzy number x, the set

$$[x]_\alpha = \{u \in \mathbb{R} : x(u) \geq \alpha\}$$

is called a α-*level set* of x. A fuzzy number x is said to be *convex* if r, s, $t \in \mathbb{R}$, $r \leq s \leq t$, implies

$$\min\{x(r), x(t)\} \leq x(s).$$

A fuzzy number x is said to be *normal* if there exists a point $u \in \mathbb{R}$ such that $x(u) = 1$. If a fuzzy number x is upper semicontinuous, convex and normal, then the α-level set of x is a closed interval $[a_\alpha, b_\alpha]$, that is,

$$[x]_\alpha = [a_\alpha, b_\alpha], \quad \alpha \in (0,1],$$

where the values $a_\alpha = -\infty$ and $b_\alpha = \infty$ are admissible. A fuzzy number x is said to be *nonnegative* if $x(u) = 0$ for all $u < 0$. The fuzzy number $\bar{\theta}$ is defined by $\bar{\theta}(u) = 1$ for $u = 0$ and $\bar{\theta}(u) = 0$ for $u \neq 0$.

Throughout this section, we denote by G the set of all nonnegative upper semicontinuous normal convex fuzzy numbers and we always assume that $L,\ R : [0,1] \times [0,1] \to [0,1]$ are two functions such that they are nondecreasing in both arguments, symmetric and $L(0,0) = 0,\ R(1,1) = 1$.

Let X be a nonempty set and $d : X \times X \to G$ be a mapping. Denote

$$(3.1) \qquad [d(x,y)]_\alpha = [\lambda_\alpha(x,y),\ \rho_\alpha(x,y)],\ x,\ y \in X, \alpha \in (0,1],$$

where $[d(x,y)]_\alpha$ is the α-level set of a fuzzy number $d(x,y) \in G$, which is actually a closed interval of $\mathbb{R}$ and $\lambda_\alpha(x,y)$, $\rho_\alpha(x,y)$ are the left and right end points of the closed interval $[d(x,y)]_\alpha$, respectively.

DEFINITION 3.1. [9] The quadruple (X, d, L, R) is called a *fuzzy metric space* if the mapping $d : X \times X \to G$ satisfies the following conditions:

(FM-1) $d(x,y) = \bar{\theta}$ if and only $x = y$,
(FM-2) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(FM-3) For any $x, y, z \in X$,
  (i) $d(x,y)(s+t) \geq L(d(x,z)(s), d(z,y)(t))$ whenever $s \leq \lambda_1(x,z)$, $t \leq \lambda_1(z,y)$ and $s + t \leq \lambda_1(x,y)$,
  (ii) $d(x,y)(s+t) \leq R(d(x,z)(s), d(z,y)(t))$ whenever $s \geq \lambda_1(x,z)$, $t \geq \lambda_1(z,y)$ and $s + t \geq \lambda_1(x,y)$.

REMARK 3.1. By Theorem 3.2 in [9], we know that if (X, d, L, R) is a fuzzy metric space with $\lim_{a \to 0^+} R(a,a) = 0$, then there exists a topology $\mathcal{T}_d$ on X such that $(X, \mathcal{T}_d)$ is a Hausdorff topological space and

$$\mathcal{U}(x) = \{U_x(\epsilon, \alpha) : \epsilon > 0, \alpha \in (0,1]\}, \quad x \in X,$$

is a basis of neighborhoods of the point x for the topology $\mathcal{T}_d$, where

$$U_x(\epsilon, \alpha) = \{y \in X : \rho_\alpha(x,y) < \epsilon\},$$

and $\rho_\alpha(x,y)$ is the right end point of $[d(x,y)]_\alpha$ defined by (3.1).

PROPOSITION 3.1. [2] *Let (X, d, L, R) be a fuzzy metric space with*

$$\lim_{a\to 0+} R(a,a) = 0, \quad \lim_{t\to\infty} d(x,y)(t) = 0, \quad x,\ y \in X, \tag{3.2}$$

then $(X, \rho_\alpha : \alpha \in (0,1])$ is a generating space of quasi-metric family and the topology $\mathcal{T}_{\{\rho_\alpha\}}$ induced by the family $\{\rho_\alpha\}$ coincides with the fuzzy topology $\mathcal{T}_d$ on (X, d, L, R), where $[d(x,y)]_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)]$ is defined by (3.1).

Proof. Since $\lim_{t\to\infty} d(x,y)(t) = 0$, it follows that $\rho_\alpha(x,y) < \infty$ for all $\alpha \in (0,1]$. Next we prove that $(X, \rho_\alpha : \alpha \in (0,1])$ is a generating space of quasi-metric family. It is obvious that $(X, \rho_\alpha : \alpha \in (0,1])$ satisfies the conditions (QM-1), (QM-2) and (QM-4) in Definition 2.1. Now we prove that it also satisfies the condition (QM-3). By the assumption that $\lim_{a\to 0^+} R(a,a) = 0$, for any $\alpha \in (0,1]$, there exists an $\mu \in (0,\alpha]$ such that $R(\mu,\mu) < \alpha$. For any given $x, y, z \in X$, let

$$\rho_\mu(x,z) = s, \quad \rho_\mu(z,y) = t.$$

By the definition of ρ_μ, it is easy to show that $s \ge \lambda_1(x,z)$ and $t \ge \lambda_1(z,y)$.

(i) If $s + t \ge \lambda_1(x,y)$, then for any $\epsilon > 0$ it follows from (FM-3)(ii) that

$$\begin{aligned} d(x,y)(s+t+2\epsilon) &\le R(d(x,z)(s+\epsilon), d(z,y)(t+\epsilon)) \\ &\le R(\mu,\mu) < \alpha. \end{aligned}$$

Hence we have $\rho_\alpha(x,y) < 2\epsilon + s + t$. By the arbitrariness of ϵ, we obtain

$$\rho_\alpha(x,y) \le s + t = \rho_\mu(x,z) + \rho_\mu(z,y). \tag{3.3}$$

(ii) If $s + t < \lambda_1(x,y)$ and $u = \lambda_1(x,y) - (s+t)$, then we have

$$\begin{aligned} 1 &= d(x,y)(\lambda_1(x,y)) = d(x,y)(u+s+t) \\ &\le R(d(x,z)(s+\tfrac{1}{2}u), d(z,y)(t+\tfrac{1}{2}u)) \\ &\le R(\mu,\mu) < \alpha, \end{aligned}$$

which is a contradiction. Therefore, the case (ii) can not happen. This proves that $(X, \rho_\alpha : \alpha \in (0,1])$ satisfies the condition (QM-3).

Besides, by Remark 3.1, the topology $\mathcal{T}_{\{\rho_\alpha\}}$ on the generating space of quasi-metric family $(X, \rho_\alpha : \alpha \in (0,1])$ coincides with the fuzzy topology $\mathcal{T}_d$ on the fuzzy metric space (X, d, L, R). This completes the proof.

From Theorem 2.1 and Proposition 3.1, we can obtain the following:

THEOREM 3.1. *Let (X_i, d_i, L, R), $i = 1, 2$, be two complete fuzzy metric spaces with*

$$\lim_{t\to\infty} d_i(x,y)(t) = 0,\ x,\ y \in X_i,\ i = 1,\ 2, \quad \text{and} \quad \lim_{a\to 0+} R(a,a) = 0.$$

Let $f : X_1 \to X_2$ be a closed mapping, $T : X_1 \to X_1$ a continuous mapping satisfying $\rho_{1\alpha}(Tx, Ty) \leq \rho_{1\alpha}(x, Ty)$ and $\rho_{2\alpha}(f(Tx), f(Ty)) \leq \rho_{2\alpha}(f(x), f(Ty))$ for every $x,\ y \in X_1$ and $\alpha \in (0,1]$, where $\rho_{i\alpha}(x,y)$ are the right end points of $[d_i(x,y)]_\alpha$ defined by (3.1), $\psi : \mathbb{R} \to \mathbb{R}$ a nondecreasing continuous function, bounded from below, and $\phi : f(X_1) \to \mathbb{R}$ a lower semicontinuous function, bounded from below. Assume that for any $u \in X_1$ with $\inf_{x\in X_1} \psi(\phi(f(x))) < \psi(\phi(f(u)))$, there exists $v \in X_1$ with $v \neq Tu$ and

$$\begin{aligned} \max\{&\rho_{1\alpha}(v, Tu) + \rho_{1\alpha}(Tu, Tv), \\ &c[\rho_{2\alpha}(f(v), f(Tu)) + \rho_{2\alpha}(f(Tu), f(Tv))]\} \\ &\leq k(\alpha)[\psi(\phi(f(u))) - \psi(\phi(f(v)))] \end{aligned}$$

for any $\alpha \in (0,1]$, where c is a given constant. Then there exists an $x_0 \in X_1$ such that

$$\inf_{x\in X_1} \psi(\phi(f(x))) = \psi(\phi(f(x_0))).$$

COROLLARY 3.2. *Let (X_i, d_i, L, R), $i = 1, 2$, be two complete fuzzy metric spaces with*

$$\lim_{t\to\infty} d_i(x,y)(t) = 0,\ x,\ y \in X_i,\ i = 1,\ 2, \quad \text{and} \quad \lim_{a\to 0+} R(a,a) = 0.$$

Let $f : X_1 \to X_2$ be a closed mapping, $T : X_1 \to X_1$ a continuous mapping satisfying $\rho_{1\alpha}(Tx, Ty) \le \rho_{1\alpha}(x, Ty)$ and $\rho_{2\alpha}(f(Tx), f(Ty)) \le \rho_{2\alpha}(f(x), f(Ty))$ for every $x,\ y \in X_1$ and $\alpha \in (0, 1]$, where $\rho_{i\alpha}(x, y)$ are the right end points of $[d_i(x, y)]_\alpha$ defined by (3.1), and $\phi : f(X_1) \to \mathbb{R}$ a lower semicontinuous function, bounded from below. Assume that for any $u \in X_1$ with $\inf_{x\in X_1} \phi(f(x)) < \phi(f(u))$, there exists $v \in X_1$ with $v \ne Tu$ and

$$\begin{aligned}\max\{&\rho_{1\alpha}(v, Tu) + \rho_{1\alpha}(Tu, Tv),\\ &c[\rho_{2\alpha}(f(v), f(Tu)) + \rho_{2\alpha}(f(Tu), f(Tv))]\}\\ &\le k(\alpha)[\phi(f(u)) - \phi(f(v))]\end{aligned}$$

for any $\alpha \in (0, 1]$, where c is a given constant. Then there exists an $x_0 \in X_1$ such that

$$\inf_{x\in X_1} \phi(f(x)) = \phi(f(x_0)).$$

Proof. The result follows from Theorem 3.1 with $\psi = I$.

From Theorem 2.4 and Proposition 3.1, we also have the following:

THEOREM 3.3. *Let (X_i, d_i, L, R), $i = 1,\ 2$, be two complete fuzzy metric spaces with*

$$\lim_{t\to\infty} d_i(x, y)(t) = 0,\ x,\ y \in X_i,\ i = 1,\ 2, \quad \text{and} \quad \lim_{a\to 0^+} R(a, a) = 0.$$

Let $f : X_1 \to X_2$ be a closed mapping, $T : X_1 \to X_1$ a continuous mapping satisfying $\rho_{1\alpha}(Tx, Ty) \le \rho_{1\alpha}(x, Ty)$ and $\rho_{2\alpha}(f(Tx), f(Ty)) \le \rho_{2\alpha}(f(x), f(Ty))$ for every $x,\ y \in X_1$ and $\alpha \in (0, 1]$, where $\rho_{i\alpha}(x, y)$ are the right end points of $[d_i(x, y)]_\alpha$ defined by (3.1), $\psi : \mathbb{R} \to \mathbb{R}$ a nondecreasing continuous function, bounded from below, and $\phi : f(X_1) \to \mathbb{R}$ a lower semicontinuous function, bounded from below. Let $S : X_1 \to X_1$ be a mapping satisfying $ST = TS$ and

$$\begin{aligned}\max\{&\rho_{1\alpha}(Sx, Tx) + \rho_{1\alpha}(Tx, TSx),\\ &c[\rho_{2\alpha}(f(Sx), f(Tx)) + \rho_{2\alpha}(f(Tx), f(TSx))]\}\\ &\le k(\alpha)[\psi(\phi(f(x))) - \psi(\phi(f(Sx)))]\end{aligned}$$

for any $x \in X_1$ and $\alpha \in (0, 1]$, where c is a given constant. Then there exists an $x_0 \in X_1$ such that $Tx_0 = x_0 = Sx_0$.

As a direct consequence, we have the following:

COROLLARY 3.4. *Let (X_i, d_i, L, R), $i = 1,\ 2$, be two complete fuzzy metric spaces with*

$$\lim_{t \to \infty} d_i(x,y)(t) = 0,\ x,\ y \in X_i,\ i = 1,\ 2, \quad \text{and} \quad \lim_{a \to 0^+} R(a,a) = 0.$$

Let $f : X_1 \to X_2$ be a mapping, and $\phi : f(X_1) \to \mathbb{R}$ a lower semicontinuous function, bounded from below and $S : X_1 \to X_1$ a mapping such that

$$\max\{\rho_{1\alpha}(Sx,x), c\rho_{2\alpha}(f(Sx), f(x))\} \le k(\alpha)[\phi(f(x)) - \phi(f(Sx))]$$

for any $x \in X_1$ and $\alpha \in (0,1]$, where c is a given constant and $\rho_{i\alpha}(x,y)$ are the right end points of $[d_i(x,y)]_\alpha$ defined by (3.1). Then there exists an $x_0 \in X_1$ such that $Sx_0 = x_0$.

REMARK 3.1. (1) Theorem 2 in [8] is a special case of Theorem 3.1 with $R = \max$, $T = I$, $\psi = I$, $c = 1$ and $k(\alpha) \equiv 1$

(2) If we take $T = I$ in Corollary 3.2, we can obtain Corollary 5.3 in [2].

(3) Corollary 3.4 generalizes Theorem 3 in [8].

(4) Corollary 3.3 in [7] is also a special case of Corollary 3.4 with $R = \max$.

4. Versions in Probabilistic Metric Spaces

In this section, we give the corresponding results in probabilistic metric spaces.

Throughout this section, we denote by $\mathcal{D}$ the set of all left continuous distribution functions.

A function $\triangle : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if the following conditions are satisfied:

(TN-1) $\triangle(a,b) = \triangle(b,a)$,

(TN-2) $\triangle(a,1) = a$,

(TN-3) $\triangle(a, \triangle(b,c)) = \triangle(\triangle(a,b), c)$,

(TN-4) $\triangle(a,b) \le \triangle(c,d)$ for $a \le c$ and $b \le d$.

DEFINITION 4.1. [12] A triple $(X, F, \triangle)$ is called a *Menger probabilistic metric space* (briefly, a Menger PM-space) if X is a nonempty set, $\triangle$ is a t-norm and $F : X \times X \to \mathcal{D}$ is a mapping satisfying the following conditions (we shall denote $F(x, y)$ by $F_{x,y}$):

(PM-1) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,
(PM-2) $F_{x,y}(0) = 0$,
(PM-3) $F_{x,y} = F_{y,x}$,
(PM-4) $F_{x,y}(s+t) \geq \triangle(F_{x,z}(s), F_{z,y}(t))$ for all $x,\ y,\ z \in X,\ s,\ t \geq 0$.

REMARK 4.1. It is pointed out in [12] that if $\triangle$ satisfies the condition $\sup_{t<1}\triangle(t,t) = 1$, then there exists a topology $\mathcal{T}$ on X such that $(X, \mathcal{T})$ is a Hausdorff topological space and the family of sets

$$\mathcal{U}(p) = \{U_p(\epsilon, \lambda) : \epsilon > 0, \lambda \in (0, 1]\}, \quad p \in X,$$

is a basis of neighborhoods of the point p for $\mathcal{T}$, where

$$U_p(\epsilon, \lambda) = \{x \in X : F_{x,p}(\epsilon) > 1 - \lambda\}.$$

Usually, the topology $\mathcal{T}$ is called (ϵ, λ)-topology on $(X, F, \triangle)$.

PROPOSITION 4.1. [2] *Let $(X, F, \triangle)$ be a Menger probabilistic metric space with a t-norm $\triangle$ satisfying the condition:*

$$\sup_{t<1} \triangle(t,t) = 1. \tag{4.1}$$

For any $\alpha \in (0, 1]$, we define $d_\alpha : X \times X \to \mathbb{R}^+$ as follows:

$$d_\alpha(x, y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \alpha\}. \tag{4.2}$$

Then (i) $(X, d_\alpha : \alpha \in (0, 1])$ is a generating space of quasi-metric family,
(ii) the topology $\mathcal{T}_{\{d_\alpha\}}$ on $(X, d_\alpha : \alpha \in (0, 1])$ coincides with the (ϵ, λ)-topology $\mathcal{T}$ on $(X, F, \triangle)$.

Proof. (i) From the definition of $\{d_\alpha : \alpha \in (0, 1]\}$, it is easy to see that $\{d_\alpha : \alpha \in (0, 1]\}$ satisfies the conditions (QM-1) and (QM-2) in Definition 2.1. Besides, it follows clearly that d_α is nonincreasing in α.

Next we prove that d_α is left continuous in α. In fact, for any given $\alpha_1 \in (0,1]$ and $\epsilon > 0$, from the definition of d_α, there exists a $t_1 > 0$ such that $t_1 < d_{\alpha_1}(x,y) + \epsilon$ and $F_{x,y}(t_1) > 1 - \alpha_1$. Letting $\delta = F_{x,y}(t_1) - (1-\alpha_1) > 0$ and $\lambda \in (\alpha_1 - \delta, \alpha_1]$, we have

$$1 - \alpha_1 < 1 - \lambda < 1 - (\alpha_1 - \delta) = F_{x,y}(t_1),$$

which implies that $t_1 \in \{t > 0 : F_{x,y}(t) > 1 - \lambda\}$. Hence we have

$$\begin{aligned} d_{\alpha_1}(x,y) &\le d_\lambda(x,y) = \inf\{t > 0 : F_{x,y}(t) > 1 - \lambda\} \\ &\le t_1 < d_{\alpha_1}(x,y) + \epsilon, \end{aligned}$$

which shows that d_α is left continuous in α.

Finally, we prove that $(X, d_\alpha, : \alpha \in (0,1])$ also satisfies the condition (QM-3).

By the condition (4.1), for any given $\alpha \in (0,1]$, there exists a $\mu \in (0,\alpha]$ such that

$$\triangle(1-\mu, 1-\mu) > 1 - \alpha.$$

Letting $d_\mu(x,z) = \sigma$ and $d_\mu(z,y) = \beta$, from (4.2), for any given $\epsilon > 0$, we have

$$F_{x,z}(\sigma + \epsilon) > 1 - \mu, \quad F_{z,y}(\beta + \epsilon) > 1 - \mu$$

and so

$$\begin{aligned} F_{x,y}(\sigma + \beta + 2\epsilon) &\ge \triangle(F_{x,z}(\sigma+\epsilon), F_{z,y}(\beta+\epsilon)) \\ &\ge \triangle(1-\mu, 1-\mu) > 1 - \alpha. \end{aligned}$$

Hence we have

$$d_\alpha(x,y) \le \sigma + \beta + 2\epsilon = d_\mu(x,z) + d_\mu(z,y) + 2\epsilon.$$

By the arbitrariness of $\epsilon > 0$, we have

$$d_\alpha(x,y) \le d_\mu(x,z) + d_\mu(z,y).$$

(ii) To prove the conclusion (ii), it is enough to prove that for any $\epsilon > 0$ and $\alpha \in (0,1]$,

$$d_\alpha(x,y) < \epsilon \quad \text{if and only if} \quad F_{x,y}(\epsilon) > 1 - \alpha.$$

In fact, if $d_\alpha(x,y) < \epsilon$, from (4.2), we have $F_{x,y}(\epsilon - \mu) > 1 - \alpha$.

Conversely, if $F_{x,y}(\epsilon) > 1 - \alpha$, since $F_{x,y}$ is a left continuous distribution function, then there exists a $\mu > 0$ such that $F_{x,y}(\epsilon - \mu) > 1 - \alpha$, and so $d_\alpha(x,y) \le \epsilon - \mu < \epsilon$. This completes the proof.

From Theorem 2.1 and Proposition 4.1, we can obtain the following:

THEOREM 4.1. *Let $(X_i, F_i, \triangle_i)$, $i = 1, 2$, be two complete Menger probabilistic metric spaces with t-norms $\triangle_i$ satisfying the condition (4.1). Let $f : X_1 \to X_2$ be a closed mapping, $T : X_1 \to X_1$ a continuous mapping satisfying*

$$\inf\{t > 0 : F_{1Tx,Ty}(t) > 1 - \alpha\} \le \inf\{t > 0 : F_{1x,Ty}(t) > 1 - \alpha\}$$

and

$$\inf\{t > 0 : F_{2f(Tx),f(Ty)}(t) > 1-\alpha\} \le \inf\{t > 0 : F_{2f(x),f(Ty)}(t) > 1-\alpha\}$$

for every x, $y \in X_1$ and $\alpha \in (0, 1]$, $\psi : \mathbb{R} \to \mathbb{R}$ a nondecreasing continuous function, bounded from below, and $\phi : f(X_1) \to \mathbb{R}$ a lower semicontinuous function, bounded from below. Assume that for any $u \in X_1$ with $\inf_{x \in X_1} \psi(\phi(f(x))) < \psi(\phi(f(u)))$, there exists a $v \in X_1$ with $v \neq Tu$ and

$$\begin{aligned} \max\{&\inf\{t > 0 : F_{1v,Tu}(t) > 1 - \alpha\} \\ &+ \inf\{t > 0 : F_{1Tu,Tv}(t) > 1 - \alpha\}, \\ &c[\inf\{t > 0 : F_{2f(v),f(Tu)}(t) > 1 - \alpha\} \\ &+ \inf\{t > 0 : F_{2f(Tu),f(Tv)}(t) > 1 - \alpha\}]\} \\ \le\ & k(\alpha)[\psi(\phi(f(u))) - \psi(\phi(f(v)))], \quad \alpha \in (0, 1], \end{aligned}$$

where $c > 0$ is a given constant. Then there exists an $x_0 \in X_1$ such that

$$\inf_{x \in X_1} \psi(\phi(f(x))) = \psi(\phi(f(x_0))).$$

From Theorem 2.4 and Proposition 4.1, we obtain the following:

THEOREM 4.2. *Let $(X_i, F_i, \triangle_i)$, $i = 1, 2$, be two complete Menger probabilistic metric spaces with t-norms $\triangle_i$ satisfying the condition (4.1). Let $f : X_1 \to X_2$ be a closed mapping, $T : X_1 \to X_1$ a continuous mapping satisfying*

$$\inf\{t > 0 : F_{1Tx,Ty}(t) > 1 - \alpha\} \le \inf\{t > 0 : F_{1x,Ty}(t) > 1 - \alpha\}$$

and

$$\inf\{t > 0 : F_{2f(Tx),f(Ty)}(t) > 1-\alpha\}$$
$$\leq \inf\{t > 0 : F_{2f(x),f(Ty)}(t) > 1-\alpha\}$$

for every x, $y \in X_1$ *and* $\alpha \in (0,1]$, $\psi : \mathbb{R} \to \mathbb{R}$ *a nondecreasing continuous function, bounded from below, and* $\phi : f(X_1) \to \mathbb{R}$ *a lower semicontinuous function, bounded from below. Let* $S : X_1 \to X_1$ *be a mapping satisfying* $ST = TS$ *and*

$$\max\{\inf\{t > 0 : F_{1Sx,Tx}(t) > 1-\alpha\}$$
$$+ \inf\{t > 0 : F_{1Tx,TSx}(t) > 1-\alpha\},$$
$$c[\inf\{t > 0 : F_{2f(Sx),f(Tx)}(t) > 1-\alpha\}$$
$$+ \inf\{t > 0 : F_{2f(Tx),f(TSx)}(t) > 1-\alpha\}]\}$$
$$\leq k(\alpha)[\psi(\phi(f(x))) - \psi(\phi(f(Sx)))], \quad \alpha \in (0,1],$$

for any $x \in X_1$, *where* $c > 0$ *is a given constant. Then there exists an* $x_0 \in X_1$ *such that* $Tx_0 = x_0 = Sx_0$.

As a consequence, we have the following :

COROLLARY 4.3. *Let* $(X_i, F_i, \triangle_i)$, $i = 1, 2$, *be two complete Menger probabilistic metric spaces with t-norms* $\triangle_i$ *satisfying the condition* (4.1), $f : X_1 \to X_2$ *a closed mapping, and* $\phi : f(X_1) \to \mathbb{R}$ *a lower semicontinuous function, bounded from below. Let* $S : X_1 \to X_1$ *be a mapping satisfying*

$$\max\{\inf\{t > 0 : F_{1Sx,x}(t) > 1-\alpha\},$$
$$c\inf\{t > 0 : F_{2f(Sx),f(x)}(t) > 1-\alpha\}\}$$
$$\leq k(\alpha)[\phi(f(x)) - \phi(f(Sx))], \quad \alpha \in (0,1],$$

for any $x \in X$, *where* $c > 0$ *is a given constant. Then there exists an* $x_0 \in X_1$ *such that* $Sx_0 = x_0$.

REMARK 4.2. (1) The corresponding results in [2, 8] are special cases of Theorem 4.1 with $\triangle = \min$, $T = I$ and $\psi = I$.

(2) Corollary 4.3 improves upon Theorem 8 in [8]

(3) If we take $X_1 = X_2$, $f = I$, $\triangle = \min$, $c = 1$ and $k(\alpha) \equiv 1$ in Corollary 4.3, then we can also obtain the corresponding results in [6, 8].

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