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TRANSLATION INVARIANT AND POSITIVE DEFINITE BILINEAR FOURIER HYPERFUNCTIONS

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§1. Introduction

It is well known in the theory of distributions and proved in [GS, S] that

(i) (Bochner–Schwartz) Every positive definite (tempered) distribution is the Fourier transform of a positive tempered measure μ.

(ii) (Schwartz kernel theorem) Let $B(\varphi, \psi)$ be a bilinear distribution. Then for some $u \in \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}^n)$ $B(\varphi, \psi) = u(\varphi(x)\overline{\psi}(y))$ for every φ, $\psi \in C_c^\infty$.

(iii) A translation invariant positive definite bilinear distribution $B(\varphi, \psi)$ is of the form $B(\varphi, \psi) = \int \widehat{\varphi}(x)\overline{\widehat{\psi}}(x)\, d\mu(x)$ for every φ, $\psi \in C_c^\infty(\mathbb{R}^n)$, where μ is a positive tempered measure.

Recall that a generalized function u is said to be *positive* if $u(\varphi) \geq 0$ for any nonnegative test function φ and is said to be *positive definite* (or of *positive type* in Schwartz[S]) if $u(\varphi * \tilde{\varphi}) \geq 0$ for any nonnegative test function φ, where $\tilde{\varphi}(x) = \overline{\varphi(-x)}$. Also, a positive measure μ is said to be *tempered* if for some $p \geq 0$ $\int (1 + |x|^2)^{-p}\, d\mu < \infty$.

A bilinear functional $B(\varphi, \psi)$ on a space Φ of test functions is called a *bilinear* generalized function if

(i) for every fixed $\psi \in \Phi$, $B(\varphi, \psi)$ is a generalized function in φ,

(ii) for every fixed $\varphi \in \Phi$, $\overline{B(\varphi, \psi)}$ is a generalized function in ψ,

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and is called *translation invariant* if $B(\varphi(x), \psi(x)) = B(\varphi(x+h), \psi(x+h))$ for all $h \in \mathbb{R}^n$. Also, a bilinear functional $B(\varphi, \psi)$ is called *positive definite* if $B(\varphi, \varphi) \geq 0$ for all $\varphi \in \Phi$.

For the case of Fourier hyperfunctions the parallel results of the above Bochner–Schwartz theorem and Schwartz kernel theorem were proved respectively in [CK, CKL].

In this paper the above property (iii) stated above will be generalized to the case of Fourier hyperfunctions (see Theorem 3.3).

§2. Preliminaries for Fourier hyperfunctions

We briefly introduce the real version of the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions as in [KCK].

DEFINITION 2.1 ([KCK]). (i) We denote by $\mathcal{F}$ the set of all infinitely differentiable functions φ in $\mathbb{R}^n$ with the property that there exist positive constants k and h such that

$$\sup_{\substack{x\in\mathbb{R}^n\\ \alpha\in\mathbb{Z}_+^n}} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty,$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha \in \mathbb{Z}_+^n$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\partial_j = \partial/\partial x_j$.
(ii) We say that $\varphi_j \to 0$ in $\mathcal{F}$ if there exist $k > 0$ and $h > 0$ such that

$$\sup_{\substack{x\in\mathbb{R}^n\\ \alpha\in\mathbb{Z}_+^n}} \frac{|\partial^\alpha \varphi_j(x)| \exp k|x|}{h^{|\alpha|}\alpha!} \to 0 \quad \text{as} \quad j \to \infty$$

(iii) We denote by $\mathcal{F}'$ the strong dual space of $\mathcal{F}$ and call its elements *Fourier hyperfunctions*.

To prove the main theorem we need the following Bochner–Schwartz theorem and Schwartz kernel theorem as in [CK, CKL].

THEOREM 2.2([CK]). *Let u be a positive definite Fourier hyperfunction. Then u is the Fourier transform of an infra-exponentially tempered measure. Conversely if μ is an infra-exponentially tempered measure then μ defines a Fourier hyperfunction.*

THEOREM 2.3([CKL]). *If* $K \in \mathcal{F}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *then a linear map* $\mathcal{K}$ *determined by*

$$(2.1) \qquad \langle \mathcal{K}\varphi, \psi \rangle = K(\psi \otimes \varphi), \quad \psi \in \mathcal{F}(\mathbb{R}^{n_1}),\ \varphi \in \mathcal{F}(\mathbb{R}^{n_2})$$

is continuous in the sense that $\mathcal{K}\varphi_j$ *converges to 0 in* $\mathcal{F}'(\mathbb{R}^{n_1})$ *if* φ_j *converges to 0 in* $\mathcal{F}(\mathbb{R}^{n_2})$. *Conversely, for every such linear map* $\mathcal{K}$ *there is one and only one Fourier hyperfunction* K *such that* (2.1) *is valid.*

The above theorems play an essential role in proving the main theorem in the next section.

§3. Main theorem

In this section we will prove the main theorem. For this we need the following lemmas which are not so trivial to prove.

LEMMA 3.1. *If* $u \in \mathcal{F}'(\mathbb{R})$ *and* $du/dx = 0$ *then* u *is a constant, i.e.,* $u(\varphi) = \int c\varphi dx$ *for some* $c \in \mathbb{R}$. *More generally, if* $u \in \mathcal{F}'(\mathbb{R}^{n-1} \times \mathbb{R})$ *and* $du/dx_n = 0$, *then* u *is of the form*

$$u(\varphi) = \tilde{u}(\int \varphi(\cdot, x_n)\, dx_n)$$

where $\tilde{u} \in \mathcal{F}'(\mathbb{R}^{n-1})$ *and* $x = (x', x_n)$.

Proof. Let $\varphi \in \mathcal{F}(\mathbb{R})$ and $\int \varphi\, dx = 0$. Then we will show that φ is of the form $\varphi(x) = d\psi/dx$ for some $\psi \in \mathcal{F}(\mathbb{R})$. Hence it follows that

$$u(\varphi) = u\left(\frac{d\psi}{dx}\right) = -\frac{du}{dx}(\psi) = 0.$$

Define $\psi(x) = \int_{-\infty}^{x} \varphi(t)\, dt$. We claim that $\psi(x) \in \mathcal{F}(\mathbb{R})$. For some $h \geq 1$ and $k > 0$ we have

$$\sup_{\substack{x\in\mathbb{R} \\ \alpha\geq 1}} \frac{|\partial^\alpha \psi(x)| \exp k|x|}{h^\alpha \alpha!} \leq \sup_{\substack{x\in\mathbb{R} \\ \alpha\geq 1}} \frac{|\partial^{\alpha-1} \varphi(x)| \exp k|x|}{h^{\alpha-1}(\alpha - 1)!} < \infty.$$

Thus it suffices to prove that

$$\sup_{x\in\mathbb{R}} |\psi(x)| \exp k|x| < \infty$$

for some $k > 0$. For $x \le 0$ we have for some $k > 0$

$$\begin{aligned} |\psi(x)| &= \int_{-\infty}^{x} \exp(2k|t|)|\varphi(t)| \exp(-k|t|) \exp(-k|t|)\, dt \\ &\le C \exp(-k|x|) \int_{-\infty}^{x} \exp(-k|t|)\, dt \\ &\le C' \exp(-k|x|). \end{aligned}$$

By the assumption that $\int \varphi(x)\, dx = 0$ we have for $x \ge 0$

$$\begin{aligned} |\psi(x)| = &= |\int_{x}^{\infty} \varphi(t)\, dt| \\ &\le \int_{x}^{\infty} \exp(2k|t|)|\varphi(t)| \exp(-k|t|) \exp(-k|t|)\, dt \\ &\le C \exp(-k|x|) \int_{x}^{\infty} \exp(-k|t|)\, dt \\ &\le C' \exp(-k|x|). \end{aligned}$$

Therefore, we have $\varphi(x) = d\psi/dx$ for some $\psi \in \mathcal{F}(\mathbb{R})$.

Now, choose $\chi \in \mathcal{F}(\mathbb{R})$ such that $\int \chi\, dx = 1$. For any $\varphi \in \mathcal{F}(\mathbb{R})$ set $\tilde{\varphi} = \varphi - (\int \varphi\, dx)\chi$. Then $\tilde{\varphi}$ belongs to $\mathcal{F}(\mathbb{R})$ and $\int \tilde{\varphi}\, dx = \int \varphi\, dx - (\int \varphi dx)(\int \chi\, dx) = 0$. Setting $u(\chi) = C$ we have $u(\varphi) = (\int \varphi\, dx)u(\chi) = c \int \varphi\, dx$.

To prove for the general case, let $u \in \mathcal{F}'(\mathbb{R}^{n-1} \times \mathbb{R})$ and χ as above. Define $\tilde{u} \in \mathcal{F}'(\mathbb{R}^{n-1})$ by

$$\tilde{u}(\tilde{\varphi}) = u(\tilde{\varphi}(x')\chi(x_n)), \quad \tilde{\varphi} \in \mathcal{F}(\mathbb{R}^{n-1}). \tag{3.1}$$

The definition (3.1) makes sense and $\tilde{u} \in \mathcal{F}'(\mathbb{R}^{n-1})$. For any $\varphi \in \mathcal{F}(\mathbb{R}^{n-1} \times \mathbb{R})$, set $\varphi_1 = \varphi - (\int \varphi(\cdot, x_n)\, dx_n)\chi(x_n)$. Then $\varphi_1 \in \mathcal{F}(\mathbb{R}^{n-1} \times \mathbb{R})$ and $u(\varphi_1) = 0$, since $\int \varphi_1(x)\, dx_n = 0$. Thus we have $u(\varphi) = u(\int \varphi(\cdot, x_n)\, dx_n \chi(x_n)) = \tilde{u}(\int \varphi(\cdot, x_n)\, dx_n)$ by the definition of u. This completes the proof.

Also, we need the following lemma which was proved in [CKL].

LEMMA 3.2 [CKL]. *Let $u \in \mathcal{F}'(\mathbb{R}^n \times \mathbb{R}^n)$. If $u(\varphi(x)\psi(y)) = 0$ for all $\varphi,\ \psi \in \mathcal{F}(\mathbb{R}^n)$ then $u \equiv 0$ in $\mathcal{F}'(\mathbb{R}^n \times \mathbb{R}^n)$.*

We are now in a position to state and prove the main theorem.

THEOREM 3.3. *Let $B(\varphi, \psi)$ be a translation invariant positive definite bilinear Fourier hyperfunction. Then $B(\varphi, \psi)$ can be written in the form*

$$B(\varphi, \psi) = \int \widehat{\varphi}(x)\overline{\widehat{\psi}}(x)\, d\mu(x), \quad \varphi,\ \psi \in \mathcal{F}$$

where μ is a positive infra-exponentially tempered measure, i.e., $\int \exp(-\epsilon|x|)\, d\mu(x) < \infty$ for every $\epsilon > 0$.

Proof. By Theorem 2.3 $B(\varphi, \psi)$ can be written as $B(\varphi, \psi) = u(\varphi(x)\overline{\psi}(y))$ for some $u \in \mathcal{F}'(\mathbb{R}^n \times \mathbb{R}^n)$.

Set, for some $h \in \mathbb{R}^n$, $u_h(\varphi(x, y)) = u(\varphi(x + h, y + h))$. Then we have

$$\begin{aligned}(u_h - u)(\varphi(x)\overline{\psi}(y)) &= u_h(\varphi(x)\overline{\psi}(y)) - u(\varphi(x)\overline{\psi}(y)) \\ &= u(\varphi(x + h)\overline{\psi}(y + h)) - u(\varphi(x)\overline{\psi}(y)) \\ &= B(\varphi(x + h), \psi(y + h)) - B(\varphi(x), \psi(y)) \\ &= 0.\end{aligned}$$

It follows from Lemma 3.2 that $u = u_h$ in $\mathcal{F}'(\mathbb{R}^n \times \mathbb{R}^n)$, i.e.,

$$u(\varphi(x, y)) = u(\varphi(x + h, y + h)), \quad h \in \mathbb{R}^n \tag{3.2}$$

Define $u_1 \in \mathcal{F}'(\mathbb{R}^n \times \mathbb{R}^n)$ by $u_1(\psi(x, y)) = u(\psi(x + y, x - y))$. Then it follows from (3.2) that u_1 is invariant under translation by h. Consider the Newton quotient

$$\varphi_{h_j} = \frac{\varphi(x_1, \cdots, x_j + h_j, \cdots, x_n, y_1 \cdots, y_n) - \varphi(x_1, \cdots, x_n, y_1 \cdots y_n)}{h_j}.$$

Since u_1 is invariant under translation by h, we have $u_1(\varphi_{h_j}) = 0$ for all $h_j \neq 0$. Letting $h_j \to 0$ we have $du_1/dx_j = 0$ for all $j = 1, 2, \ldots, n$. Applying Lemma 3.1 for the variables $x_1, \cdots x_n$, separately, we have

$u_1(\varphi(x,y)) = \tilde{u}(\int \varphi(x,y)dx)$ for some $\tilde{u} \in \mathcal{F}'(\mathbb{R}^n)$. Thus for any $\varphi \in \mathcal{F}(\mathbb{R}^n \times \mathbb{R}^n)$ we have

$$\begin{aligned} u(\varphi(x,y)) &= u_1\left(\varphi\left(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x - \frac{1}{2}y\right)\right) \\ &= \tilde{u}\left(\int \varphi\left(\frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}x - \frac{1}{2}y\right) dx\right) \\ &= 2^n \tilde{u}(\int \varphi(x, x-y)dx). \end{aligned}$$

Consequently, it follows that

$$B(\varphi,\psi) = u(\varphi(x)\overline{\psi}(y)) = \tilde{u}(\int \varphi(x)\overline{\psi}(x-y)\,dx) = \tilde{u}(\varphi * \psi^*).$$

Since $B(\varphi,\psi)$ is positive definite, $\tilde{u}$ is a positive definite Fourier hyperfunction. By Theorem 2.2 we have

$$\tilde{u}(\varphi * \psi^*) = \int (\varphi * \psi^*)^\frown \, d\mu(x) = \int \widehat{\varphi}(x)\overline{\widehat{\psi}}(x)\, d\mu(x)$$

for some positive measure μ which is infra-exponentially tempered. This completes the proof.

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