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A CROFTON STYLE FORMULA AND ITS APPLICATION ON THE UNIT SPHERE S^{2*}

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1. Introduction

Crofton's formula on Euclidean plane E^2 states: Let Γ be a rectifiable curve of length L and let G be a straight line. Then

$$\int_{G\cap\Gamma\neq\phi} n\, dG = 2L$$

where n is the number of the intersection points of G with the curve Γ.

L. A. Santaló gave a generalization of Crofton's formulas to the sphere and found integral formulas in [6, 8] and R. Howard and H. Tasaki obtained formulas in Riemannian homogeneous spaces in [4] and [9], respectively.

In this paper, we define strips on S^2 and their density and, using them, we obtain integral formulas which have relation to the strips. We see that formula (7) can be regarded as a generalization of the Crofton's formula. We also obtain some inequalities on the unit sphere as their applications.

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2. Preliminaries and Notations

A *circle* on the sphere is defined to be a plane section of a sphere. A *great circle* on the sphere is the section which the plane passes through the center of the sphere; a small circle on the sphere is the section of the other case.

The element of area on the unit sphere will be represented by $d\Omega$. So if θ and ϕ are the spherical coordinates of the point Ω, then

$$d\Omega = \sin\theta d\theta \wedge d\phi. \tag{1}$$

A non-directed great circle C on the unit sphere can be determined by one of its poles, that is, by one of the extremities of the diameter perpendicular to it. Since $d\Omega$ is the element of area of one of these extremities, the "density" for measuring sets of great circles on the sphere is

$$dC = d\Omega \tag{2}$$

that is, the "measure" of a set of great circles on the sphere is defined as the integral of (2) extended over this set.

DEFINITION 1. A closed curve on the sphere is said to be *convex* when it cannot be cut by a great circle in more than two points.

A convex curve divides the sphere into two parts, one of which is always wholly contained in a hemisphere; that is, there is always a great circle which has the whole convex curve on the same side; we only have to consider, for example, a great circle tangent to the curve at some point.

When we say a *convex set*, we understand that part of the surface of the sphere which is limited by a convex curve and is smaller than or equal to a hemisphere.

3. Integral Formulas and Some Inequalities on S^2

W. Blaschke states Crofton's formula in [1]: Let Γ be a convex curve of length L on the unit sphere and let C be a great circle. Then the measure of the great circles which cut a convex curve Γ is equal to the length of this curve, that is,

$$\int_{C\cap\Gamma\neq\phi} dC = L. \tag{3}$$

If K is a closed convex curve on the unit sphere of enclosing area F and length L and $F(K_\rho)$ and $L(K_\rho)$ are the area and the length of the outer parallel curve K_ρ to K at the distance $\rho \leq \pi/2$, respectively, then [2]

$$\begin{aligned} F(K_\rho) &= L\sin\rho + F\cos\rho + 2\pi(1 - \cos\rho), \\ L(K_\rho) &= L\cos\rho + (2\pi - F)\sin\rho. \end{aligned} \tag{4}$$

As an application of the formula (4) L. A. Santaló proved the isoperimetric inequality on the unit sphere: If K is a closed convex curve on the unit sphere of enclosing area F and length L, then

$$L^2 + F^2 - 4\pi F \geq 0. \tag{5}$$

Assume $\rho \leq \pi$. By a *strip* B of breadth ρ we mean the closed part of the sphere consisting of all points that lie between two parallel circles at a distance $\rho/2$ from a great circle.

The position of a strip B can be determined by the position of its mid-parallel great circle; in other words, it can be determined by the pole Ω of the great circle. Therefore the density for sets of strips of fixed breadth will be

$$dB = d\Omega \tag{6}$$

where Ω is the extremity of the great circle.

Now we get a generalization of the Crofton's formula on S^2.

THEOREM 1. *Let K be a convex set on the unit sphere of area F and perimeter L and of the greatest radius r_M of spherical curvature of ∂K and let B be the strip of breadth $\rho\,(0 \leq \rho \leq \pi - 2r_M)$. If K is fixed and B is moving, then*

$$\int_{B\cap K\neq\phi} dB = L\cos\frac{\rho}{2} + (2\pi - F)\sin\frac{\rho}{2}. \tag{7}$$

Proof. If $B \cap K \neq \phi$, the mid-parallel C of B intersects the parallel set $K_{\frac{\rho}{2}}$ of K in the distance $\frac{\rho}{2}$. Conversely, if the mid-parallel C of B intersects $K_{\frac{\rho}{2}}$, then B intersects K. Since the parallel set $K_{\frac{\rho}{2}}$ is convex, using (3) and (4), we have

$$\int_{B\cap K\neq\phi} dB = \int_{C\cap K_{\frac{\rho}{2}}\neq\phi} dC = L\cos\frac{\rho}{2} + (2\pi - F)\sin\frac{\rho}{2}. \quad \square$$

REMARK 1. If $\rho = 0$, then the strip reduces to a great circle and so Theorem 1 implies the Crofton's formula as a special case.

As a particular case of a set K in Theorem 1, (7) gives us the following.

COROLLARY 1. *The measure of all strips of breadth ρ that contain a fixed point P is* $2\pi\sin\frac{\rho}{2}$.

Proof. Since the point has area zero and the perimeter zero, the proof follows from (7). $\square$

COROLLARY 2. *Let N convex sets $K_i\,(i = 1,\cdots,N)$ be contained in a bounded convex set K on the unit sphere and let L_i be the perimeter of K_i and let r_M be the greatest of the radii r_{M_i} of the spherical curvature of ∂K_i. If N convex sets $K_i\,(i = 1,\cdots,N)$ are fixed and B is moving, then for the strip B of breadth $\rho(0 \leq \rho \leq \pi - 2r_M)$* we have

$$\int_{B\cap K\neq\phi} n\,dB = \sum_{1}^{N} L_i\cos\frac{\rho}{2} + (2\pi N - \sum_{1}^{N} F_i)\sin\frac{\rho}{2},$$

where n denotes the number of the sets K_i that are intersected by the strip B.

Proof. For the strip B of breadth ρ ($0 \leq \rho \leq \pi - 2r_M$), by Theorem 1, we have

$$\begin{aligned}\int_{B\cap K\neq\phi} n\,dB &= \sum_1^N m(B; B\cap K_i \neq \phi)\\ &= \sum_1^N \left(L_i \cos\frac{\rho}{2} + (2\pi - F_i)\sin\frac{\rho}{2}\right)\\ &= \sum_1^N L_i \cos\frac{\rho}{2} + \left(2\pi N - \sum_1^N F_i\right)\sin\frac{\rho}{2}. \quad \square\end{aligned}$$

THEOREM 2. *Let D be the domain on the unit sphere, not necessarily convex, of area F and let B be the strip of breadth ρ. If D is fixed and B is moving, then*

$$\int_{B\cap D\neq\phi} f\,dB = 2\pi F \sin\frac{\rho}{2},$$

where f is the area of $B \cap D$.

Proof. The density for sets of pairs of points and strips (Ω, B), assuming the independence of Ω and B, is $d\Omega \wedge dB$. The measure of the set of pairs (Ω, B) such that $\Omega \in B \cap D$ is

$$\int_{\Omega\in B\cap D} d\Omega \wedge dB.$$

To calculate this integral we fix Ω and apply Corollary 1. Then

$$\begin{aligned}\int_{\Omega\in B\cap D} d\Omega \wedge dB &= \int_{\Omega\in D} d\Omega \int_{\Omega\in B} dB\\ &= 2\pi \sin\frac{\rho}{2} \int_{\Omega\in D} d\Omega\\ &= 2\pi F \sin\frac{\rho}{2},\end{aligned}$$

where ρ is the breadth of B. On the other hand, if we fix B and call f the area of $B \cap D$, then

$$m(\Omega, B;\, \Omega \in B \cap D) = \int_{B \cap D \neq \phi} f\, dB.$$

Thus

$$\int_{B \cap D \neq \phi} f\, dB = 2\pi F \sin\frac{\rho}{2}. \quad \square$$

THEOREM 3. *Let K be a closed convex curve on the unit sphere of enclosing area F and length L. If K is fixed and B is moving, then for the strip B with the breadth $\rho(\rho \leq \pi)$*

$$\int_{B \cap K \neq \phi} (\bar{u}^2 + \bar{f}^2) dB \geq 8\pi^2 F \sin\frac{\rho}{2},$$

where $\bar{u}$, $\bar{f}$ are the perimeter and area of the convex hull of $B \cap K$, respectively.

Proof. Consider the convex hull $\overline{B \cap K}$ of $B \cap K$ and let $\bar{u}$, $\bar{f}$ be the perimeter and area of $\overline{B \cap K}$, respectively. Then, by (5), we have $\bar{u}^2 + \bar{f}^2 \geq 4\pi \bar{f}$. Since $f < \bar{f}$, using Theorem 2, we have

$$\begin{aligned} \int_{B \cap K \neq \phi} (\bar{u}^2 + \bar{f}^2) dB &\geq 4\pi \int_{B \cap K \neq \phi} \bar{f} dB \\ &\geq 4\pi \int_{B \cap K \neq \phi} f dB = 8\pi^2 F \sin\frac{\rho}{2}. \quad \square \end{aligned}$$

The following lemma is due to L. A. Santaló.

LEMMA 1. *Let K be a convex curve on the unit sphere of enclosing area F and length L with the maximum breadth $\delta\,(\delta \leq \pi/2)$. Then*

$$L/(2\pi - F) \leq \tan\frac{\delta}{2}.$$

Proof. See [7]. $\square$

THEOREM 4. *Let K be a closed convex curve on the unit sphere of enclosing area F and length L with the maximum breadth δ ($\delta \le \frac{\pi}{2}$) and let r_M be the greatest radius of spherical curvature of ∂K. Then for any number ρ in $[0, \pi - 2r_M]$, we have*

$$L \cos \frac{\rho}{2} + (2\pi - 3F) \sin \frac{\rho}{2} \ge 0. \tag{8}$$

Proof. Let B be the strip with the breadth $\rho\,(0 \le \rho \le \pi - 2r_M)$.

Consider the convex hull $\overline{B \cap K}$ of $B \cap K$ and let $\bar{u}$, $\bar{f}$ be the perimeter and area of $\overline{B \cap K}$, respectively.

Since the diameter δ' of $\overline{B \cap K}$ is also less than or equal to $\frac{\pi}{2}$, we have $\tan \frac{\delta'}{2} \le 1$ and so, by Lemma 1, we have $\bar{u} \le 2\pi - \bar{f}$.

Using the inequality $\bar{u}^2 + \bar{f}^2 \le (\bar{u} + \bar{f})^2$, by Theorem 1 and Theorem 3 we have

$$\begin{aligned} 8\pi^2 F \sin \frac{\rho}{2} &\le \int_{B \cap K \neq \phi} (\bar{u}^2 + \bar{f}^2) dB \le \int_{B \cap K \neq \phi} (\bar{u} + \bar{f})^2 dB \\ &\le 4\pi^2 \int_{B \cap K \neq \phi} dB = 4\pi^2 \left(L \cos \frac{\rho}{2} + (2\pi - F) \sin \frac{\rho}{2} \right). \end{aligned}$$

Hence we have

$$L \cos \frac{\rho}{2} + (2\pi - 3F) \sin \frac{\rho}{2} \ge 0. \quad \Box$$

The followings justify the our main theorem for a region on the unit sphere.

REMARK 2. (1) Let K be a circle of radius $\frac{\pi}{4}$. Then $r_M = \frac{\pi}{4}$, $L = \sqrt{2}\pi$ and $F = (2 - \sqrt{2})\pi$. So if we take $\rho = \frac{\pi}{3}$, then

$$L \cos \frac{\rho}{2} + (2\pi - 3F) \sin \frac{\rho}{2} = 1.34\pi > 0.$$

(2) If K is a circle of radius $\frac{7}{18}\pi$ and we take $\rho = \frac{\pi}{2}$, then the inequality (8) in Theorem 4 fails to hold. Indeed, in this case $L = 5.90$ and $F = 4.13$ and so

$$L \cos \frac{\rho}{2} + (2\pi - 3F) \sin \frac{\rho}{2} = -0.15 < 0.$$

So our assumption in Theorem 4 is needed.

COROLLARY 3. *Let K be a convex set on the unit sphere of area F and perimeter L with the diameter less than or equal to $\frac{\pi}{2}$ and with the greatest radius of spherical curvature of ∂K is less than or equal to $\frac{\pi}{4}$. Then*

$$L + (2\pi - 3F) \geq 0.$$

Proof. Take $\rho = \frac{\pi}{2}$ in Theorem 4. Then the proof follows from the Theorem 4 immediately. □

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