Bull. Korean Math. Soc. **33** (1996), No. 4, pp. 531–536

THE GENERATOR OF THE ANALYTIC GROUP WITH ITS LIE ALGEBRA $\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{sl}(2, \mathbb{F})$

MI-AENG WI

1. Introduction

Let $\mathbb{F}$ denote $\mathbb{R}$ or $\mathbb{C}$. Put $A = SL(2, \mathbb{F})$. Define $\mathbb{P}(\mathbb{F}^2)$ to be the set of all 1-dimensional subspaces of $\mathbb{F}^2$. Then the natural action of A on $\mathbb{F}^2$ induces an action on $\mathbb{P}(\mathbb{F}^2)$.

REMARK. The action on $\mathbb{P}(\mathbb{F}^2)$ is doubly transitive with the kernel $\{\pm I\}$. In particular, $\mathrm{PSL}(2, \mathbb{F})$ acts faithfully on $\mathbb{P}(\mathbb{F}^2)$

NOTATION. $G = \langle A, B\rangle$ means that G is generated by A and B.
$Z(G)$ is a center of G.
$[\ ,\ ]$ is a commutator.

Let $v = \langle (0, 1)\rangle$ and B be a stabilizer of v in A.
Thus $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid 0 \neq a,\ \ b \in \mathbb{F} \right\}$. Put $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{F} \right\}$.
Let $\exp : \mathfrak{sl}(2, \mathbb{F}) \to A$ be the exponential map. Let $\mathfrak{s}_0$ be the subalgebra of $\mathfrak{sl}(2, \mathbb{F})$ given by $\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{F} \right\}$. Then we have the following Lemmas:

LEMMA 1.1. $[B, B] = U = \exp(\mathfrak{s}_0)$.

Proof. Since $B/U \cong \mathbb{F} - \{0\}$ is abelian, $[B, B] \leq U$.

Conversely, for $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in B$, $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in U$,
$\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right] = \begin{pmatrix} 1 & t(1 - a^{-2}) \\ 0 & 1 \end{pmatrix}$. Thus, $U \leq [B, B]$. Also,

Received March 10, 1996.
1991 AMS Subject Classification: 22E30.
Key words and phrases: Analytic group, doubly transitive, acts faithfully, semisimple, levi decomposition, covering group.

$U = \exp(\mathfrak{s}_0)$ since the exponential map from $\mathfrak{sl}(2,\mathbb{F})$ to A is given by ordinary exponential matrices. Put

$$B^{opp} = \left\{ \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \mid 0 \neq a,\ b \in \mathbb{F} \right\} \text{ and } U^{opp} = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \mid c \in \mathbb{F} \right\}.$$

LEMMA 1.2. *A is generated by U and U^{opp}.*

Proof. We will use Gaussian Elimination.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be in $SL(2,\mathbb{F})$. Put $K = \langle U, U^{opp} \rangle$ and put $\lambda(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $\mu(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $t \in \mathbb{F}$. Left multiplication by $\mu(t)$ induces the elementary row operation of adding t times first row to the second row. Left multiplication by $\lambda(t)$ induces the elementary row operation of adding t times second row to the first row. Also, left multiplication by τ interchanges rows and negates the second row. We first note that $\tau \in K$, since $\tau = \lambda(1)\mu(-1)\lambda(1)$. Next, the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ is in K since $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \mu(1)\lambda(a)\mu(-a^{-1})\lambda(a^2+a)(\tau^3)$. Next, any matrix $\begin{pmatrix} 0 & b \\ -b^{-1} & d \end{pmatrix}$ is in K, since $\begin{pmatrix} 0 & b \\ -b^{-1} & d \end{pmatrix} = \tau^{-1}\lambda(-db^{-1})\begin{pmatrix} -b & 0 \\ 0 & -b^{-1} \end{pmatrix}$.

Finally, if $a \neq 0$, we have $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu(-ca^{-1})^{-1}\lambda(ab)\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$ is in K.

COROLLARY 1.3. *Any two conjugates of U generate $A = SL(2,\mathbb{F})$.*

Proof. Let $x, y \in A$ with $U^x \neq U^y$. By Lemma 1.1, $U^x = [B^x, B^x]$ and $U^y = [B^y, B^y]$. So, $B^x \neq B^y$. By doubly transitivity of A on $P(\mathbb{F}^2)$, there is $h \in A$ such that $B^{xh} = B$, $B^{yh} = B^{opp}$. Then $\langle U^x, U^y \rangle = \langle U^{xh}, U^{yh} \rangle^{h^{-1}} = \langle U, U^{opp} \rangle^{h^{-1}} = A^{h^{-1}} = A$ by Lemma 1.2.

PROPOSITION 1.4. *Let G^* be an analytic group with $L(G^*) = \mathfrak{sl}(2,\mathbb{F})$. Put $\mathfrak{s}_0$ = the subalgebra of $\mathfrak{sl}(2,\mathbb{F})$ given by* $\left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{F} \right\}$.

Then G^* is generated by any two conjugates of $\exp^*(\mathfrak{s}_0)$, where $\exp^* : \mathfrak{sl}(2,\mathbb{F}) \to G^*$ is the exponential map.

Proof. Put $U^* = \exp^*(\mathfrak{s}_o)$. Let X^*, Y^* be two distinct conjugates of U^* in G^*.

Put $H^* = \langle X^*, Y^* \rangle$. We will show that $H^* = G^*$. We have G^* semisimple. Hence $Z(G^*)$ is discrete, and any proper normal subgroup of G^* is contained in $Z(G^*)$ since $PSL(2,\mathbb{F})$ is simple for any field $\mathbb{F}$ of order bigger then 3. Then $G^*/Z(G^*)$ is a simple analytic group with Lie algebra $\mathfrak{sl}(2,\mathbb{F})$. Thus $G^*/Z(G^*) \cong PSL(2,\mathbb{F})$, and the quotient map $\varphi : G^* \to G^*/Z(G^*)$ is a covering of $PSL(2,\mathbb{F})$. Then we have a commutative diagram; for any $g \in G^*$,

$$\begin{array}{cccc} & G* & \xrightarrow{ad_g} & G* \\ \exp * & & & \\ \mathfrak{sl}(2,\mathbb{F}) & \varphi\downarrow & & \varphi\downarrow \\ \exp & & & \\ & PSL(2,\mathbb{F}) & \xrightarrow{ad_{\varphi(g)}} & PSL(2,\mathbb{F}) \end{array}$$

Here $X^* = (\exp^*(\mathfrak{s}_o))^g$ for $g \in G^*$.

Then we have $\varphi(X^*) = (\exp(\mathfrak{s}_o))^{\varphi(g)}$ by the above commutative diagram. Similarly, $\varphi(Y^*) = (\exp(\mathfrak{s}_o))^{\varphi(h)}$ for some $h \in G^*$. Suppose $\varphi(X^*) \neq \varphi(Y^*)$. Let ψ be the quotient map $SL(2,\mathbb{F}) \to PSL(2,\mathbb{F})$.

Since ψ is an epimorphism, we have $\varphi(g) = \psi(g')$ for some $g' \in SL(2,\mathbb{F})$. Then we have a commutative diagram:

$$\begin{array}{cccc} & SL(2,\mathbb{F}) & \xrightarrow{ad_{g'}} & SL(2,\mathbb{F}) \\ \exp_1 & & & \\ \mathfrak{sl}(2,\mathbb{F}) & \psi\downarrow & & \psi\downarrow \\ \exp & & & \\ & PSL(2,\mathbb{F}) & \xrightarrow{ad_{\varphi(g)}} & PSL(2,\mathbb{F}) \end{array}$$

This diagram show that $\varphi(X^*) = \psi(\exp_1(\mathfrak{s}_o)^{g'})$. Put $\tilde{X} = \exp_1(\mathfrak{s}_o)^{g'}$. Similarly, put $\tilde{Y} = \exp_1(\mathfrak{s}_o)^{h'}$, where $\varphi(h) = \psi(h')$. Then $\varphi(X^*) = \psi(\tilde{X})$ and $\varphi(Y^*) = \psi(\tilde{Y})$. Thus $\psi(\tilde{X}) \neq \psi(\tilde{Y})$ and so $\tilde{X} \neq \tilde{Y}$. But $\langle \tilde{X}, \tilde{Y} \rangle = SL(2, \mathbb{F})$ by Corollary 1.3. So, $\langle \varphi(X^*), \varphi(Y^*) \rangle = \psi \langle \tilde{X}, \tilde{Y} \rangle = PSL(2, \mathbb{F})$. Now, then $\varphi(H^*) = PSL(2, \mathbb{F})$. But Ker $\varphi = Z(G^*)$ and so $G^* = H^* Z(G^*)$. Since G^* is semisimple, $G^* = [G^*, G^*] = H^*$. So, we are done in this case.

It remains to show that $\varphi(X^*) \neq \varphi(Y^*)$. Suppose $\varphi(X^*) = \varphi(Y^*)$. Then $Z(G^*)X^* = Z(G^*)Y^*$. But $X^* = (Z(G^*)X^*)^\circ$, a connected component of 1 in $Z(G^*)X^*$, since X^* is connected and so $X^* \leq (Z(G^*)X^*)^\circ$. Also $Z(G^*)X^*/X^* \cong Z(G^*)/(Z(G^*) \cap X^*)$ discrete. Hence $X^* = (Z(G^*)X^*)^\circ$. Similarly, $Y^* = (Z(G^*)Y^*)^\circ$. Hence $X^* = Y^*$, a contradiction.

2. Main Hypothesis

PART I : Assume that G is an analytic group over $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$. Let $\mathfrak{g}$ be the Lie algebra of G. Assume that $\mathfrak{g} = \text{rad}(\mathfrak{g}) \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{sl}(2, \mathbb{F})$.

Before stating Part II of the hypothesis, we first establish noation, as follows.

Let $\mathfrak{s}_o$ be the subalgebra of $\mathfrak{sl}(2, \mathbb{F})$ given by $\left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbb{F} \right\}$.

$$\begin{aligned} \mathfrak{q} &= \text{nil rad}(\mathfrak{g}) \\ \mathfrak{s} &= \mathfrak{q} \oplus \mathfrak{s}_o \\ S_0 &= \exp(\mathfrak{s}_o) \\ S &= \exp(\mathfrak{s}) \\ Q &= \exp(\mathfrak{q}) \\ M &= \exp(\mathfrak{m}) \end{aligned}$$

PART II: Let X denote the group of all continuous automorphisms of S. Assume that no non-identity X-invariant subgroup of S is normal in G.

LEMMA 2.1. [Theorem 3.18.13 in [3]] *Let G be an analytic group with Lie algebra $\mathfrak{g}$, and Q(resp. N) the radical (resp. nil radical) of G. Then Q and N are closed. Suppose that $\mathfrak{g} = \mathfrak{q} + \mathfrak{m}$ is a Levi decomposition of $\mathfrak{g}$ and that M is the analytic subgroup of G defined by m. Then $G = QM$, and M is a maximal semisimple analytic subgroup of G.*

REMARK. Notice that $\mathfrak{g} = \text{rad}(\mathfrak{g}) \oplus \mathfrak{m}$ is a Levi decomposition of $\mathfrak{g}$. Then, by Lemma 2.1, we have $G = RM$, where M is a maximal semisimple connected subgroup of G and R is the radical of G. Also, Q is a connected normal Lie subgroup of G and S is connected nilpotent.

Now, we want to describe what are the relations among S, Q, M and G under the main hypothesis:

LEMMA 2.2. [Proposition 2.2 in [5]] $G = QM$ *and* $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{m}$.

LEMMA 2.3. [Lemma 4.3 in [5]] $S = QS_o$ *and* $S_o \cap Q = 1$.

Let D denote the inverse image of $Z(G/Q)$ in G, where $Z(G/Q)$ is a center of G/Q

LEMMA 2.4. $S \cap D = Q$

Proof. We have $S = QS_o$ and $S_o \cap Q = 1$ by Lemma 2.3. Thus $S \cap D = QS_o \cap D = Q(S_o \cap D) = Q$

LEMMA 2.5. [Lemma 3.2 in[5]] *M is a covering group of $PSL(2, \mathbb{F})$.*

3. Main Theorem

THEOREM 3.1. $G = \langle S, S^x \rangle$ *for any $x \in G - N_G(S)$, where $N_G(S)$ is a normaliter of S in G.*

Proof. Let $x \in G - N_G(S)$. Then $S \neq S^x$ Put $\overline{G} = G/Q$. Then $\bar{S} \neq \bar{S}^x$, since $Q \leq S \cap S^x$. Here $\bar{G} \cong M/(M \cap Q)$. Put $\bar{M} = M/(M \cap Q)$. We need to show that the canonical map $\bar{M} \to \bar{M}/Z(\bar{M})$ is a covering of $PSL(2, \mathbb{F})$, where $Z(\bar{M})$ is a center of $\bar{M}$. By Lemma 2.5, M is a covering group of $PSL(2, \mathbb{F})$ and $M/K \cong PSL(2, \mathbb{F})$, where K discrete kernel of the covering map $M \to PSL(2, \mathbb{F})$. Since M

is semisimple, $M = [M, M]$ and so $Z(M) = K$. Now, $\bar{M}/Z(\bar{M}) = M/(M \cap Q)/Z(M)/Z(M \cap Q) \cong M/Z(M) = M/K \cong PSL(2, \mathbb{F})$. Since $\bar{M}$ is semisimple, $Z(\bar{M}) = K/Q$ is discrete. Hence, the canonical map is a covering of $PSL(2, \mathbb{F})$.

Now, let $\pi; \bar{M} \to PSL(2, \mathbb{F})$ and let $\bar{M}_o$ be a subgroup of $\bar{M}$ generated by two conjugate of S_o. Since $S_o \cap \ker\pi = 1$, $S_o \ker\pi = S_o \times \ker\pi$. Also, S_o is connected, and so S_o $\ker\pi/S_o \cong \ker\pi$ discrete. So, S_o is connected component of 1 in $S_o \ker\pi$. Thus S_o is the unique conjugate of S_o contained in $S_o \ker\pi$. Thus the restriction of π to $\bar{M}_0$ is surjective by Corollary 1.3. Hence $\bar{M} = \bar{M}_0 Z(\bar{M})$. Since $\bar{M} = [\bar{M}, \bar{M}] = [\bar{M}_0 Z(\bar{M}), \bar{M}_0 Z(\bar{M})] = [\bar{M}_0, \bar{M}_0] \le \bar{M}_0, \bar{M} = \bar{M}_0$. Thus $\bar{M} = \langle S_0, S_0^x \rangle$ for $x \in G - N_G(S)$. Since $G = QM$ by Lemma 2.2, $G = Q\langle S_0, S_0^x \rangle = \langle QS_0, QS_0^x \rangle = \langle S, S^x \rangle$ for $x \in G - M_G(S)$.

COROLLARY 3.5. *$Q = S \cap S^x$ for any $x \in G - N_G(S)$.*

Proof. We have that $Q \le S \cap S^x$. Put $\bar{G} = G/Q$. Then $\bar{S} \cap \bar{S}^x$ is normal in $\bar{G} = \langle \bar{S}, \bar{S}^x \rangle$. Thus $S \cap S^x$ is normal in G. However, Q is the largest subgroup in S which is normal in G by Lemma 2.4. Thus $S \cap S^x \le Q$ and so $S \cap S^x = Q$.

References

1. N. Bourbaki, *Elements of Mathematics: Lie Groups and Lie algebras, Chapters 1-3*, Springer-Verlag, New York (1989).
2. J. E. Humphreys, *Introduction To Lie Algebras and Representation Theory*, Springer-Verlag, New York (1972).
3. V. S. Varadarajan, *Lie GRoups, Lie Algebras, and Their Representations Prentice-Hall*, New York, 1974.
4. F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Scott, Foresman and Company, Illinois, 1971.
5. Mi-Aeng Wi, *The Structure of A Connected Lie Group G with its Lie Algebra* $\mathfrak{g} = rad(\mathfrak{g}) \oplus \mathfrak{sl}(2, \mathbb{F})$, Honam Mathematical Journal **17** (1995), 7-14.

DEPTMENT OF MATHEMATICS, CHONBOOK UNIVERSITY, JEONJU 560-759, KOREA