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ODD SOLUTIONS TO PERTURBED CONSERVATION LAWS

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§1. Introduction

This paper treats the existence of odd solutions of the Cauchy problem for a perturbation of a conservation law. That is, we find a function $u = u(t,x)$ satisfying the differential equation

$$u_t + div(f(u)) + g(u)F(x) = 0 \quad \text{for } t > 0,\ x \in \mathbf{R}^n, \tag{1}$$

and the initial condition

$$u(0,x) = \phi(x) \qquad \text{for } x \in \mathbf{R}^n \tag{2}$$

This paper was inspired by the paper of M. Schonbek[8],[9] who proved existence of solutions to singular scalar conservation laws of the form

$$u_t + f(u)_x + \frac{g(u)}{|x|} = 0 \tag{3}$$

by regularizing the equation and taking a singular limit using the theory of compensated compactness. Schonbek worked on (3) for $x > 0$ with boundary condition

$$u(t,0) = 0. \tag{4}$$

We provide a semigroup approach to M. Schonbek's work, but we mainly treat the Cauchy problem (1)-(2) which is not singular at the

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origin. Section 3 deals with the singular case. M. Schonbek worked in one dimensional space, but we work in n-dimensional ones.

This paper is also influenced by Crandall's paper[4]. Crandall solved the Cauchy problem for a conservation law

$$v_t + div(f(v)) = 0 \quad \text{for } t > 0,\ x \in \mathbf{R}^n,$$

$$v(0,x) = \psi(x) \qquad \text{for } x \in \mathbf{R}^n$$

from the point of view of the theory of semigroups of nonlinear transformations.

In order to formulate the problem in an abstract form, we choose a Banach space X, namely $X = L^1(\mathbf{R}^n)$, and we regard the solution u in (1) as a map of $\mathbf{R}^+$ into X (the map $t \to u(t,\cdot)$). Let

$$Au = -\sum_{i=1}^{n}(f(u))_{x_i} \qquad \text{for } u \in \mathcal{D}(A),$$

$$Bu = -g(u)F(x) \qquad \text{for } u \in \mathcal{D}(B)$$

where $\mathcal{D}(A)$, the domain of A, and $\mathcal{D}(B)$, the domain of B, are suitable subsets of X. For a precise description of $\mathcal{D}(A)$, see [4]. Then the problem can be formally rewritten as the abstract Cauchy problem

$$\frac{du}{dt} = Au + Bu,$$

$$u(0) = u_0.$$

Let A be m-dissipative on X. Let B be (globally) Lipschitzian and dissipative on X. Then, by perturbation theory, $A + B$ on $\mathcal{D}(A+B) = \mathcal{D}(A)$ is m-dissipative on X.

§2. Basic Theorems

We consider the Cauchy problem (1)-(2) and prove the existence of solutions to this problem.

The following is Benilan's extension of Crandall's theorem[1].

PROPOSITION. *Let $f : \mathbf{R} \to \mathbf{R}^n$ be continuous with $f(0) = 0$ and $\lim_{x\to 0} \frac{|f(x)|}{|x|^{\frac{n-1}{n}}} = 0$. Then $Au = -\sum_{i=1}^n \frac{\partial}{\partial x_i}(f_i(u))$, with $\mathcal{D}(A)$ defined suitably, is m-dissipative on X.*

The assumption that $\lim_{x\to 0} \frac{|f(x)|}{|x|^{\frac{n-1}{n}}} = 0$ holds if f is differentiable at the origin. It is not a restriction (beyond continuity) at all if $n = 1$.

THEOREM 1. *Let the assumptions of the Proposition hold for A. Let $Bu(x) = -F(x)g(u)$. Suppose that $g : \mathbf{R} \to \mathbf{R}$ is (globally) Lipschitzian and nondecreasing and that $F : \mathbf{R}^n \to \mathbf{R}$ is in $L^\infty(\mathbf{R}^n)$ and nonnegative a.e. Then*

$$Cu = -\sum_{j=1}^n (f_j(u))_{x_j} - F(x)g(u)$$

with $\mathcal{D}(C) = \mathcal{D}(A)$ is m-dissipative on $X = L^1(\mathbf{R}^n)$.

Proof. To apply the perturbation theory, we show that B is (globally) Lipschitzian and dissipative on X:

$$||Bu - Bv|| \le \int_{\mathbf{R}^n} |F(x)||g(u) - g(v)|dx$$

$$\le ||F||_{L^\infty} ||g||_{Lip} ||u - v||,$$

since $F \in L^\infty(\mathbf{R}^n)$ and g is Lipschitzian. Thus B is (globally) Lipschitzian.

For each $u \in X$, define $\mathcal{J}$ by

$$\mathcal{J}(u) = \{\phi \in X^* : ||\phi||^2 = ||u||^2 =< u, \phi >\}.$$

$\mathcal{J}$ is a (multivalued) function called the *duality map* of X. For $u \in X = L^1(\mathbf{R}^n)$, $\phi(u) = ||u||sign_0 u$ is a section of the map $\mathcal{J}$ on X. For each $u, v \in \mathcal{D}(B)$,

$$\begin{aligned} &< Bu - Bv, \phi(u - v) > \\ &= - \int_{\mathbf{R}^n} F(x)[g(u(x)) - g(v(x))]||u - v||sign_0(u(x) - v(x))dx \\ &\le 0, \end{aligned}$$

since g is nondecreasing. Thus B is dissipative.

Therefore C is m-dissipative on $X = L^1(\mathbf{R}^n)$.

Now we introduce the operator G. Let G be an isomorphism on an arbitrary Banach space X. That is, $G : X \to X$ is bijective, linear and G and G^{-1} are continuous. Let $X_o = \{u \in X : Gu = u\}$ be the fixed point set of G. Then X_o is a closed subset of X.

EXAMPLE 1. Let $X = L^1(\mathbf{R}^n)$ and $(G_o u)(x) = -u(-x)$ for $x \in \mathbf{R}^n$ and $u \in X$. Then $X_o = \{u \in X : u \text{ is odd in } X\}$

2. Let $X = L^1(\mathbf{R}^n)$ and $(G_e u)(x) = u(-x)$ for $x \in \mathbf{R}^n$ and $u \in X$. Then the corresponding X_o is $\{u \in X : u \text{ is even in } X\}$.

Both choices of X_o are closed subspaces in this example.

G commutes with A means that for $u \in \mathcal{D}(A)$, $Gu \in \mathcal{D}(A)$ and $G(Au) = A(Gu)$ and G anticommutes with A means that for $u \in \mathcal{D}$, $Gu \in \mathcal{D}(A)$ and $G(Au) = -A(Gu)$.

LEMMA 1. *Let A, B be as in Section 1. Let $G = G_o$ or G_e; $G_o u(x) = -u(-x)$ or $G_e u(x) = u(-x)$. Then A commutes with G_o if each f_i is odd. A always anticommutes with G_e. Similarly, B commutes with G_o provided g is odd and F is even.*

Proof. For $u \in \mathcal{D}(A)$,

$$(AG_o(u))(x) = A(-u(-x)) = -\sum_i [f_i(-u(-x))]_{x_i},$$

if each f_i is odd

$$= \sum_i [f_i(u(-x))]_{x_i} = -(Au)(-x) = (G_o A(u))(x).$$

That is, $G_o(Au)(x) = A(G_o u)(x)$, for each $x \in \mathbf{R}^n$ and each $u \in \mathcal{D}(A)$.

The other commuting assertions are proved in a similar way.

THEOREM 2. *Let the assumptions of Theorem 1 hold. Suppose that each f_i is odd for $1 \leq i \leq n$, g is odd and F is even. Let $X_o = \{u \in$*

$X : u$ is odd in $X\}$. Then for $\lambda > 0$, $(I - \lambda(A + B)) : \mathcal{D}(A) \cap X_o \to X_o$ is onto Here $\mathcal{D}(A)$ is as in [1].

Proof. Consider

$$u - \lambda(A + B)u = k$$

for $k \in X$ and $\lambda > 0$. Applying $G = G_o$ to both sides,

$$Gu - \lambda(A + B)Gu = Gk, \tag{5}$$

because G commutes with both A and B. Then $I - \lambda(A + B)$ maps $\mathcal{D}(A) \cap X_o$ into X_o, since $u = Gu$ implies $k = Gk$. By m-dissipativity

$$u - \lambda(A + B)u = k$$

has a unique solution u. For $k \in X_o$,i.e.$Gk = k$, Gu is also a solution from (5). By uniqueness, $u = Gu$. Therefore $\mathcal{R}[(I - \lambda(A + B))|_{X_o}] = X_o$ for $\lambda > 0$.

By the Crandall-Liggett theorem[5], $A + B$ determines a strongly continuous contraction semigroup on X_o. In other words, if $T(t)\phi = \lim_{n\to\infty}(I - \frac{t}{n}(A + B))^{-n}\phi$ for $\phi \in X$, then for $\phi \in X_o$, $T(t)\phi \in X_o$ for $t \geq 0$ and the semigroup T acts on X_o. See [3] for an analogous idea in the Hamilton-Jacobi context.

§3. An Illustration of the Singular Case

We consider the Cauchy problem (1)-(2) with Dirichlet boundary condition at 0.

Of concern is when $F(0) = \infty$ and $F(x) \leq M_\delta < \infty$ if $|x| \geq \delta > 0$. See (3) for a specific example.

For simplicity we work in one dimensional space, so consider

$$u_t + f(u)_x + \frac{g(u)}{h(x)} = 0 \quad \text{for } x \in \mathbf{R},\ t \geq 0 \tag{6}$$

with the initial condition

$$u(0, x) = \phi(x) \qquad \text{for } x \in \mathbf{R} \tag{7}$$

and with boundary condition

$$u(t,0) = 0 \qquad \text{for } t \geq 0. \tag{8}$$

where h is continuous, $h(x) \to 0$ as $x \to 0$ and for each $\delta > 0$, there is a $\epsilon > 0$ such that $h(x) > \epsilon$ if $x \geq \delta$. We also assume h is even and $\int_0^1 \frac{1}{h(x)}dx < \infty$.

We consider

$$u_t - (A + B_n)u = 0 \quad \text{for } x \in \mathbf{R}, t \geq 0.$$

where $(A+B_n)u = -f(u)_x - \frac{g(u)}{h_n(x)} = C_n u$ and $\frac{1}{h_n(x)}$ is an even function in $L^\infty(\mathbf{R}) \cap C(0,\infty)$ and nonnegative a.e. and $\{h_n\}$ is a decreasing sequence converging to h.

Let $Cu = -f(u)_x - \frac{g(u)}{h(x)}$. We want to solve

$$u - Cu = l \tag{9}$$

where l is in a dense set of the space L^2_{odd}, for example, in $C_c^\infty(\mathbf{R}/\{0\})$ and $suppl \subset [\epsilon, \frac{1}{\epsilon}] \cup [-\frac{1}{\epsilon}, -\epsilon]$ and l is odd. We know that C_n is m-dissipative by Theorem 1.

LEMMA 2. *Let the assumptions of Theorem 2 hold when we take* $F(x) = \frac{1}{h_n(x)}$ *and let* u_n *satisfy*

$$u_n - \lambda(A + B_n)u_n = l$$

where $l \in C_c^\infty(\mathbf{R}/\{0\})$ *and* $suppl \subset [\epsilon, \frac{1}{\epsilon}] \cup [-\frac{1}{\epsilon}, -\epsilon]$ *and* l *is odd. Then* $u_n \in L^p(\mathbf{R})$ *and*

$$||u_n||_{L^p} \leq ||l||_{L^p}$$

for $1 \leq p \leq \infty$.

Proof. Since C_n is m-dissipative by Theorem 1,

$$u_n + f(u_n)_x + \frac{g(u_n)}{h_n(x)} = l \tag{10}$$

has a unique solution u_n. Multiplying (10) by $|u_n|^{p-2}u_n = |u_n|^{p-1}$ $\text{sign}_0(u_n)$ and integrating both sides yields

$$\int_{-\infty}^{\infty} |u_n|^p dx + \int_{-\infty}^{\infty} f(u_n)_x |u_n|^{p-2} u_n dx + \int_{-\infty}^{\infty} \frac{g(u_n)}{h_n(x)} |u_n|^{p-2} u_n dx$$
$$= \int_{-\infty}^{\infty} l |u_n|^{p-2} u_n dx.$$

The second term of the left side is

$$\int_{-\infty}^{\infty} f(u_n)_x |u_n|^{p-2} u_n dx = \int_{-\infty}^{\infty} (u_n)_x f'(u_n) |u_n|^{p-2} u_n dx$$
$$= \int_{-\infty}^{\infty} H(u_n)_x dx = 0,$$

where $H' = G$ and $G(u_n) = f'(u_n)|u_n|^{p-2}u_n$.

The third term of the left side is

$$\int_{-\infty}^{\infty} \frac{g(u_n)}{h_n(x)} |u_n|^{p-2} u_n \geq 0,$$

since g is odd and nondecreasing. Thus

$$||u_n||_{L^p}^p \leq \int_{-\infty}^{\infty} |l||u_n|^{p-1} \leq ||l||_{L^p} ||u_n||_{L^p}^{p-1}$$

by Hölder's inequality. Therefore $||u_n||_{L^p} \leq ||l||_{L^p}$ for $1 \leq p < \infty$,so for $1 \leq p \leq \infty$.

THEOREM 3. *Let the assumptions of Theorem 2 hold. Suppose also that h is even, $h \in L^\infty(\mathbf{R})$, $\frac{1}{h} \in L^1(0,1)$ and for each $\epsilon > 0$, h is bounded away from zero on (ϵ, ∞). Let l be as in Lemma 2, and let u_n be the unique solution of*

$$u_n - \lambda(A + B_n)u_n = l, \tag{11}$$

where $\lambda > 0$ is fixed. Then $u_n \to u$ in $L^1(\mathbf{R})$ and u solves (9). Thus $A + B = C$ (with $\mathcal{D}(C) = \mathcal{D}(A)$) is densely defined and essentially m-dissipative and so determines a contraction semigroup on $L^1_{odd}(\mathbf{R})$.

Proof. The last part follows by the Crandall-Liggett theorem. Let u_n be the unique solution of (11). We need only show that $\{u_n\}$ is a Cauchy sequence in $L^1(\mathbf{R})$. For $n > m$,

$$u_n + \lambda(f(u_n))_x + \frac{g(u_n)}{h_n(x)}) = l$$

implies

$$u_m + \lambda(f(u_m)_x + \frac{g(u_m)}{h_n(x)}) = \tilde{l}$$

where $\tilde{l} = \lambda g(u_m)(\frac{1}{h_n(x)} - \frac{1}{h_m(x)}) + l$.

Since $A + B$ is dissipative,

$$\begin{aligned} ||u_n - u_m||_{L^1} &\leq ||l - \tilde{l}||_{L^1} \\ &\leq \lambda \int_{-\infty}^{\infty} |g(u_m(x))||\frac{1}{h_n(x)} - \frac{1}{h_m(x)}|dx. \end{aligned}$$

But $|g(u_m(x))| \leq C_1 = ||g||_{C[-a,a]}$ where $a = ||l||_\infty$ since $||u_m||_\infty \leq ||l||_\infty$. To simplify the following calculation, suppose h is monotone nondecreasing on $[0, \delta]$ for some $\delta \geq 0$. Then ,for example,we can take

$$h_n(x) = \begin{cases} h(x) & \text{for } |x| > \frac{1}{n} \\ h(\frac{1}{n}) & \text{for } |x| \leq \frac{1}{n} \end{cases}$$

for sufficiently large n. Thus the above calculation becomes

$$\begin{aligned} ||u_n - u_m||_{L^1} &\leq 2\lambda C_1 \int_0^{\frac{1}{n}} |\frac{1}{h_n(x)} - \frac{1}{h_m(x)}|dx \\ &\leq 2\lambda C_1 \int_0^{\frac{1}{n}} \frac{1}{h_n(x)} dx \\ &\leq 2\lambda C_1 \int_0^{\frac{1}{n}} \frac{1}{h(x)} dx \to 0 \end{aligned}$$

as $n \to \infty$(with $n > m$) since $\frac{1}{h} \in L^1(0,1)$. Thus $u_n \to u$ and u is the desired solution of (9). This completes the proof.

We close with some remarks. It is not difficult to state a version of Theorem 3 which works for $x \in \mathbf{R}^n$ rather than $x \in \mathbf{R}$. But the local integrability of $\frac{1}{h}$ near the origin remains a crucial hypothesis.

If $h(x) = c|x|^\alpha$ with $c, \alpha > 0$, then Theorem 3 holds when $\alpha < 1$. Schonbek's example was for $\alpha = 1$,which is a much harder case. We hope to extend over techniques to cover this and other cases for which $\int_0^1 \frac{1}{h} dx = \infty$ in the future.

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