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ON THE HYERS–ULAM–RASSIAS STABILITY OF THE EQUATION $f(x^2 - y^2 + rxy) = f(x^2) - f(y^2) + rf(xy)$

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1. Introduction

The stability problem of functional equations has been originally raised by S. M. Ulam. In 1940, he posed the following problem: Give conditions in order for a linear mapping near an approximately additive mapping to exist (see [9]).

In 1941, this problem was solved by D. H. Hyers [3] for the first time. This problem has been further generalized and solved by Th. M. Rassias [6]. Thereafter, the stability problem of functional equations has been extended in various directions and studied by several mathematicians (see [4]).

A mapping $f : X \to Y$, where X and Y are some Banach spaces, is called a solution of the Hosszú's functional equation if and only if

$$f(x + y - xy) = f(x) + f(y) - f(xy)$$

holds for all $x, y \in X$. The only mappings $f : \mathbf{R} \to \mathbf{R}$ such that $f(x) = a(x) + b$, where $a : \mathbf{R} \to \mathbf{R}$ is an additive mapping and $b \in \mathbf{R}$, satisfy the Hosszú's equation for the case $X = Y = \mathbf{R}$ (see [2]). Since Hosszú's equation has on $\mathbf{R}$ the same solutions (up to a constant) as the additive Cauchy equation for which the Hyers-Ulam stability has been already proved by Hyers [3], it is natural to expect the Hyers-Ulam stability for the Hosszú's functional equation.

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C. Borelli [1] tried to prove the Hyers-Ulam stability for the Hosszú's functional equation. But his conditions for the stability of this equation are stronger than those of the linear functional equation (see [6] or [7]). In the same paper, he asked whether the Hyers-Ulam-Rassias stability of the Hosszú's equation is also true.

The present paper results from an attempt to answer this question of Borelli. Throughout this paper, let E_1 be a real normed space for which the multiplication '$\cdot$' between the elements is defined, and suppose $E_1 = \{x^2 : x \in E_1\} \cup \{-x^2 : x \in E_1\}$. The real space $\mathbf{R}$ is an example for the space E_1. Further, let E_2 be a real Banach space. We can now consider the following functional equation

$$(1) \qquad Vf(x,y) := f(x^2 - y^2 + rxy) - f(x^2) + f(y^2) - rf(xy) = 0$$

as a variation of the Hosszú's functional equation.

If r is a rational number in (1), every additive mapping $f : \mathbf{R} \to \mathbf{R}$ satisfies the equation (1). Moreover, each mapping of the form $a(x) + b$, where $a : \mathbf{R} \to \mathbf{R}$ is an additive mapping and $b \in \mathbf{R}$, also satisfies the given functional equation (1).

The author has proved in his paper [5] the Hyers-Ulam stability for the functional equation (1). This paper concerns a generalization of the previous result of [5], i.e., the Hyers-Ulam-Rassias stability of the functional equation (1) shall be investigated, and we shall find that our conditions for the stability of the equation (1) are similar to those of the linear functional equation (see [6]):

THEOREM 1. *Let $p < 1$, $r > 0$, $r \neq 1$, and $\theta > 0$ be given. Suppose $f : E_1 \to E_2$ to be a mapping such that*

$$(2) \qquad \|Vf(x,y)\| \leq \theta \left(\|x^2\|^p + \|y^2\|^p \right)$$

for all $x, y \in E_1$. Then there exists a unique mapping $T : E_1 \to E_2$ satisfying

$$(3) \qquad \|T(x) - f(x)\| \leq \frac{2\theta}{|r - r^p|} \|x\|^p$$

and

$$VT(x,y) = 0 \tag{4}$$

for all $x, y \in E_1$.

2. Proof of the Theorem 1

Substituting x for y in (2) we obtain

$$\|f(rx^2) - rf(x^2)\| \le 2\theta\|x^2\|^p. \tag{5}$$

We now assert that

$$\|f(r^n x^2) - r^n f(x^2)\| \le 2\theta \sum_{i=0}^{n-1} r^{n-1-i+ip}\|x^2\|^p \tag{6}$$

for all $n \in \mathbb{N}$ and any $x \in E_1$.
On account of (5), the assertion is true in the case when $n = 1$. Let the inequality (6) be true for a fixed $n \ge 1$. Replacing x by $\sqrt{r}x$ in (6), we then get

$$\|f(r^{n+1}x^2) - r^n f(rx^2)\| \le 2\theta \sum_{i=0}^{n-1} r^{n-1-i+(i+1)p}\|x^2\|^p.$$

Hence, it holds

$$\begin{aligned} &\|f(r^{n+1}x^2) - r^{n+1}f(x^2) + r^{n+1}f(x^2) - r^n f(rx^2)\| \\ &\qquad \le 2\theta \sum_{i=0}^{n-1} r^{n-1-i+(i+1)p}\|x^2\|^p, \end{aligned}$$

and it follows from (5) and the inequality just above that

$$\|f(r^{n+1}x^2) - r^{n+1}f(x^2)\| \le 2\theta \sum_{i=0}^{n} r^{n-i+ip}\|x^2\|^p.$$

This implies the validity of the assertion.
By putting $y = -x$ in (2) we have, instead of (5), the following inequality

$$\|f(-rx^2) - rf(-x^2)\| \leq 2\theta \|x^2\|^p.$$

Analogously, we can assert and easily prove

$$\|f(-r^n x^2) - r^n f(-x^2)\| \leq 2\theta \sum_{i=0}^{n-1} r^{n-1-i+ip} \|x^2\|^p \tag{7}$$

for all $n \in \mathbf{N}$ and every $x \in E_1$.
According to (6), (7) and the properties of the space E_1, we conclude

$$\|f(r^n x) - r^n f(x)\| \leq 2\theta \sum_{i=0}^{n-1} r^{n-1-i+ip} \|x\|^p \tag{8}$$

for all $n \in \mathbf{N}$ and each $x \in E_1$. Given $x \in E_1$, we now define

$$T(x) := \begin{cases} \lim_{n\to\infty} r^{-n} f(r^n x) & \text{for } r > 1, \\ \lim_{n\to\infty} r^n f\left(r^{-n} x\right) & \text{for } 0 < r < 1. \end{cases}$$

Let $n > m > 0$. Using (8), we then obtain, for $r > 1$,

$$\begin{aligned} \|r^{-n} f(r^n x) - r^{-m} f(r^m x)\| &= r^{-n} \|f(r^{n-m} r^m x) - r^{n-m} f(r^m x)\| \\ &\leq 2\theta \sum_{i=0}^{n-m-1} r^{-1-(1-p)m-(1-p)i} \|x\|^p \\ &\leq 2\theta \frac{r^{(p-1)m}}{r - r^p} \|x\|^p. \end{aligned}$$

Similarly, we have for $0 < r < 1$

$$\left\|r^n f\left(r^{-n} x\right) - r^m f\left(r^{-m} x\right)\right\| \leq 2\theta \frac{r^{(1-p)n}}{r - r^p} \|x\|^p.$$

These two inequalities just above imply that both sequences

$$\left\{r^{-n} f(r^n x)\right\}_{n\in\mathbf{N}} \quad (\text{for } r > 1)$$

and

$$\left\{r^n f\left(r^{-n} x\right)\right\}_{n\in\mathbf{N}} \quad (\text{for } 0 < r < 1)$$

are Cauchy sequences. Since the completeness of E_2 has been assumed, the value of T exists for all $x \in E_1$. From (8) it follows

$$\|T(x) - f(x)\| \leq \frac{2\theta}{|r - r^p|}\|x\|^p \tag{9}$$

for all $x \in E_1$. Now, suppose x to be an arbitrary element of E_1. Then, due to (2), we get

$$\begin{aligned}
&\|T(x^2 - y^2 + rxy) - T(x^2) + T(y^2) - rT(xy)\| = \\
&= \begin{cases} \lim_{n\to\infty} \frac{1}{r^n}\|f(r^n x^2 - r^n y^2 + r^n \cdot rxy) \\ \qquad\qquad -f(r^n x^2) + f(r^n y^2) - rf(r^n xy)\|, \\ \lim_{n\to\infty} r^n \|f\left(\frac{1}{r^n}x^2 - \frac{1}{r^n}y^2 + \frac{1}{r^n}rxy\right) \\ \qquad\qquad -f\left(\frac{1}{r^n}x^2\right) + f\left(\frac{1}{r^n}y^2\right) - rf\left(\frac{1}{r^n}xy\right)\| \end{cases} \\
&\leq \begin{cases} \lim_{n\to\infty} \theta r^{(p-1)n}(\|x^2\|^p + \|y^2\|^p) & \text{for } r > 1, \\ \lim_{n\to\infty} \theta r^{(1-p)n}(\|x^2\|^p + \|y^2\|^p) & \text{for } 0 < r < 1 \end{cases} \\
&= 0.
\end{aligned}$$

Hence, we conclude

$$T(x^2 - y^2 + rxy) = T(x^2) - T(y^2) + rT(xy) \tag{10}$$

for all $x, y \in E_1$.
Assume that $T' : E_1 \to E_2$ is a mapping satisfying (9) and (10). It immediately follows from (10) that

$$T(r^n x) = r^n T(x) \text{ and } T'(r^n x) = r^n T'(x) \tag{11}$$

for every $n \in \mathbb{N}$ and each $x \in E_1$. Further, assume $T'(x) \neq T(x)$ for some $x \in E_1$. In view of (11) and (9), we then have

$$\begin{aligned}
\|T'(x) - T(x)\| &\leq \begin{cases} r^{-n}\|T'(r^n x) - T(r^n x)\| \\ r^n\|T'(r^{-n}x) - T(r^{-n}x)\| \end{cases} \\
&\leq \begin{cases} 4\theta \frac{r^{(p-1)n}}{r - r^p}\|x\|^p & \text{for } r > 1, \\ 4\theta \frac{r^{(1-p)n}}{r^p - r}\|x\|^p & \text{for } 0 < r < 1. \end{cases}
\end{aligned}$$

The above inequality holds for every natural number n, which leads to the contradiction. Therefore, we conclude that the mapping T satisfying (9) and (10) is unique. □

Analogously, we can prove a similar theorem for the case when $p > 1$. It is the only significant difference from the proof of Theorem 1 to define

$$T(x) := \begin{cases} \lim_{n\to\infty} r^n f(r^{-n}x) & \text{for } r > 1, \\ \lim_{n\to\infty} r^{-n} f(r^n x) & \text{for } 0 < r < 1. \end{cases}$$

THEOREM 2. *Let $p > 1$, $r > 0$, $r \neq 1$, and $\theta > 0$ be given. Assume that $f : E_1 \to E_2$ is a mapping such that*

$$\|Vf(x,y)\| \leq \theta\left(\|x^2\|^p + \|y^2\|^p\right)$$

for all $x, y \in E_1$. Then there exists a unique mapping $T : E_1 \to E_2$ satisfying

$$\|T(x) - f(x)\| \leq 2\theta \frac{\|x\|^p}{|r - r^p|}$$

and

$$VT(x,y) = 0$$

for all $x, y \in E_1$.

REMARK. According to Th. M. Rassias [8], there is a continuous mapping $f : \mathbf{R} \to \mathbf{R}$ such that $|f(x+y) - f(x) - f(y)| \leq |x| + |y|$ for any $x, y \in \mathbf{R}$, and such that there exists, for every continuous additive mapping $T : \mathbf{R} \to \mathbf{R}$, some $x \in \mathbf{R}$ satisfying $|T(x) - f(x)| > |x|$. It is a counter-example for the Theorem in [6]. Can we also find such a counter-example for Theorem 1 and 2 for the case $p = 1$?

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