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THE EXISTENCE OF SOLUTIONS OF A NONLINEAR SUSPENSION BRIDGE EQUATION

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0. Introduction

In this paper we investigate a relation between the multiplicity of solutions and source terms in a nonlinear suspension bridge equation in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, under Dirichlet boundary condition

$$u_{tt} + u_{xxxx} + bu^+ = f(x) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R, \tag{0.1}$$

$$u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \tag{0.2}$$

$$u \text{ is } \pi\text{-periodic in } t \text{ and even in } x \text{ and } t, \tag{0.3}$$

where the nonlinearity $-(bu^+)$ crosses an eigenvalue λ_{10}. This equation represents a bending beam supported by cables under a load f. The constant b represents the restoring force if the cables stretch. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression.

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Let L be the differential operator, $Lu = u_{tt} + u_{xxxx}$. Then the eigenvalue problem for $u(x,t)$

$$Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times R$$

with (0.2) and (0.3), has infinitely many eigenvalues

$$\lambda_{mn} = (2n+1)^4 - 4m^2 \qquad (m, n = 0, 1, 2, \cdots)$$

and corresponding eigenfunctions $\phi_{mn}(m, n \geq 0)$ given by

$$\phi_{mn} = \cos 2mt \cos(2n+1)x.$$

We note that all eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17.$$

Let Q be the square $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and H the Hilbert space defined by

$$H = \left\{ u \in L^2(Q) : \quad u \text{ is even in } x \text{ and } t \quad \right\}.$$

Then the set of eigenfunctions $\{\phi_{mn}\}$ is an orthonormal base in H. Hence equation (0.1) with (0.2) and (0.3) is equivalent to

$$Lu + bu^+ = f \quad \text{in} \quad H.$$

In this paper we shall concern with only the case that the nonlinearity $-bu^+$ crosses an eigenvalue λ_{10}. In [3, 4, 6], the authors investigate the existence of solutions of a nonlinear suspension bridge equation (0.1), where the forcing term f is supposed to be $1 + \epsilon h$ (h is bounded) and the nonlinearity $-(bu^+)$ crosses an eigenvalue λ_{10}. Our concern is the case that f is generated by two eigenfunctions ϕ_{00} and ϕ_{10}.

It is a well known fact (cf. Theorem 1.1 of [4]) that if $f \in H$ and $-1 < b < 3$, then equation (0.1) with (0.2) and (0.3) has a unique solution.

In this paper we suppose that $3 < b < 15$ and f is generated by ϕ_{00} and ϕ_{10}. Our goal is to reveal two regions R_1, R_3 in two dimensional subspace space of the Hilbert space H spanned by ϕ_{00} and ϕ_{10} that (i) if $f \in R_1$ then (0.1) has a positive solution and (ii) if $f \in R_3$ then (0.1) has a negative solution (cf. Theorem 1.1). Finally we give a conjecture which reveals a relation between the multiplicity of solutions and source terms.

1. A Variational Reduction Method

In this section, we suppose $3 < b < 15$. Under this assumption, we have a concern with the multiplicity of solutions of a nonlinear suspension bridge equation

$$Lu + bu^{+} = f \quad \text{in} \quad H \tag{1.1}$$

Here we suppose that f is generated by two eigenfunctions ϕ_{00} and ϕ_{10}, that is, $f = s_1\phi_{00} + s_2\phi_{10}(s_1, s_2 \in \mathbb{R})$.

To study equation (1.1), we use the contraction mapping theorem to reduce the problem from an infinite dimensional one in H to a finite dimensional one.

Let V be the two dimensional subspace of H spanned by $\{\phi_{00}, \phi_{10}\}$ and W be the orthogonal complement of V in H. Let P be an orthogonal projection H onto V. Then every element $u \in H$ is expressed by

$$u = v + w,$$

where $v = Pu$, $w = (I - P)u$. Hence equation (1.1) is equivalent to a system

$$Lw + (I - P)(b(v + w)^{+}) = 0, \tag{1.2}$$

$$Lv + P(b(v + w)^{+}) = s_1\phi_{00} + s_2\phi_{10}. \tag{1.3}$$

Here we look on (1.2) and (1.3) as a system of two equations in the two unknowns v and w.

LEMMA 1.1. *For fixed $v \in V$, (1.2) has a unique solution $w = \theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous (with respect to the L^2 norm) in terms of v.*

Proof. We use the contraction mapping theorem. Let $\delta = \frac{1}{2}b$. Rewrite (1.2) as

$$(-L - \delta)w = (I - P)(b(v + w)^{+} - \delta(v + w)),$$

or equivalently,

$$w = (-L - \delta)^{-1}(I - P)g_v(w), \tag{1.4}$$

where

$$g_v(w) = b(v + w)^+ - \delta(v + w).$$

Since

$$|g_v(w_1) - g_v(w_2)| \le |b - \delta||w_1 - w_2|,$$

we have

$$\|g_v(w_1) - g_v(w_2)\| \le |b - \delta|\|w_1 - w_2\|,$$

where $\| \ \|$ is the L^2 norm in H. The operator $(-L-\delta)^{-1}(I-P)$ is a self adjoint compact linear map from $(I - P)H$ into itself. The eigenvalues of $(-L - \delta)^{-1}(I - P)$ in W are $(\lambda_{mn} - \delta)^{-1}$, where $\lambda_{mn} \le -15$ or $\lambda_{mn} \ge 17$. Therefore its L^2 norm is $\max\left\{\frac{1}{15+\delta}, \frac{1}{17-\delta}\right\}$. Since $|b - \delta| < \min\{15 - \delta, 17 + \delta\}$, it follows that for fixed $v \in V$, the right hand side of (1.4) defines a Lipschitz mapping W into itself with Lipschitz constant $\gamma < 1$. Hence, by the contraction mapping principle, for given $v \in V$, there is a unique $w \in W$ which satisfies (1.2).

Also, it follows, by the standard argument principle, that $\theta(v)$ is Lipschitz continuous (with respect to the L^2 norm) in terms of v.

By Lemma 1.1, the study of the multiplicity of solutions of (1.1) is reduced to the study of the multiplicity of solutions of an equivalent problem

$$Lv + P(b(v + \theta(v)))^+ = s_1\phi_{00} + s_2\phi_{10} \tag{1.5}$$

defined on the two dimensional subspace V spanned by $\{\phi_{00}, \phi_{10}\}$.

While one feels instinctively that (1.5) ought to be easier to solve, there is the disadvantage of an implicitly defined term $\theta(v)$ in equation (1.5). However, in our case, it turns out that we know $\theta(v)$ for special v's.

If $v \ge 0$ or $v \le 0$, then $\theta(v) \equiv 0$. For example, let us take $v \ge 0$ and $\theta(v) = 0$. Then equation (1.2) reduces to

$$L0 + (I - P)(bv^+) = 0$$

which is satisfied because $v^+ = v$ and $(I - P)v = 0$, since $v \in V$.

Since the subspace V is spanned by $\{\phi_{00}, \phi_{10}\}$, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \geq 0, |c_2| \leq c_1\}$$

so that $v \geq 0$ for all $v \in C_1$ and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_{00} + c_2\phi_{10} \mid c_1 \leq 0, |c_2| \leq |c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$.

Thus, even if we do not know $\theta(v)$ for all $v \in V$, we know $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$.

Now, we define a map $\Phi : V \to V$ given by

$$\Phi(v) = Lv + P(b(v + \theta(v))^+), \quad v \in V. \tag{1.6}$$

Then Φ is continuous on V, since θ is continuous on V and we have the following lemma.

LEMMA 1.2. *$\Phi(cv) = c\Phi(v)$ for $c \geq 0$ and $v \in V$.*

Proof. Let $c \geq 0$. If v satisfies

$$L\theta(v) + (I - P)(b(v + \theta(v))^+) = 0,$$

then

$$L(c\theta(v)) + (I - P)(b(cv + c\theta(v))^+) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore we have

$$\begin{aligned}\Phi(cv) =& L(cv) + P(b(cv + \theta(cv))^+) \\ =& L(cv) + P(b(cv + c\theta(v))^+) \\ =& c\Phi(v).\end{aligned}$$

Lemma 1.2 implies that Φ maps a cone with vertex 0 onto a cone with vertex 0. Now we investigate the image of the cones C_1 and C_3 under

Φ. First we consider the image of the cone C_1. If $v = c_1\phi_{00}+c_2\phi_{10} \geq 0$, we have

$$\begin{aligned}\Phi(v) =&L(v) + P(b(v+\theta(v))^+)\\ =&c_1\lambda_{00}\phi_{00} - c_2\lambda_{10}\phi_{10} + b(c_1\phi_{00} + c_2\phi_{10})\\ =&c_1(b+\lambda_{00})\phi_{00} + c_2(b-\lambda_{10})\phi_{10}.\end{aligned}$$

Thus the images of the rays $c_1\phi_{00} \pm c_1\phi_{10} (c_1 \geq 0)$ can be explicitly calculated and they are

$$c_1(b+\lambda_{00})\phi_{00} \pm c_1(b+\lambda_{10})\phi_{10} \qquad (c_1 \geq 0).$$

Therefore Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \geq 0, |d_2| \leq \left(\frac{b+\lambda_{10}}{b+\lambda_{00}}\right) d_1 \right\}.$$

Here the restriction $\Phi|_{C_1} : C_1 \to R_1$ is bijective. Second we consider the image of the cone C_3. If

$$v = -c_1\phi_{00} + c_2\phi_{10} \leq 0 \quad (c_1 \geq 0,\ |c_2| \leq c_1),$$

we have

$$\begin{aligned}\Phi(v) =&L(v) + P(b(v+\theta(v))^+)\\ =&Lv + P(0)\\ =&-c_1\lambda_{00}\phi_{00} + c_2\lambda_{10}\phi_{10}.\end{aligned}$$

Thus the images of the rays $-c_1\phi_{00} \pm c_1\phi_{10}$ $(c_1 \geq 0)$ can be explicitly calculated and they are

$$-c_1\lambda_{00}\phi_{00} \pm c_1\lambda_{10}\phi_{10} \qquad (c_1 \geq 0).$$

Thus Φ maps the cone C_3 onto the cone

$$R_3 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_1 \leq 0,\ d_2 \leq |\frac{\lambda_{10}}{\lambda_{00}}||d_1| \right\}.$$

Here the restriction $\Phi|_{C_1} : C_3 \to R_3$ is bijective. We note that R_1 is in the right half plane and R_3 is in the left half plane.

THEOREM 1.1. (i) *If f belongs to R_1, then equation* (1.1) *has a positive solution.*

(ii) *If f belongs to R_3, then equation* (1.1) *has a negative solution.*

Now we set

$$C_2 = \{c_1\phi_{00} + c_2\phi_{10} \mid c_2 \geq 0,\ c_2 \geq |c_1|\},$$

$$C_4 = \{c_1\phi_{00} + c_2\phi_{10} \mid c_2 \leq 0,\ c_2 \leq -|c_1|\}.$$

Then the union of C_1, C_2, C_3, C_4 is the space V.

Lemma 1.2 means that the images $\Phi(C_2)$ and $\Phi(C_4)$ are the cones in the plane V. Before we investigate the images $\Phi(C_2)$ and $\Phi(C_4)$, we set

$$R'_2 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_2 \geq 0, -|\frac{\lambda_{00}}{\lambda_{10}}|d_2 \leq d_1 \leq \left|\frac{b+\lambda_{00}}{b+\lambda_{10}}\right| d_2 \right\},$$

$$R'_4 = \left\{ d_1\phi_{00} + d_2\phi_{10} \mid d_2 \leq 0, |\frac{\lambda_{00}}{\lambda_{10}}|d_2 \leq d_1 \leq \left(\frac{b+\lambda_{00}}{b+\lambda_{10}}\right) d_2 \right\}.$$

Then the union of R_1, R'_2, R_3, R'_4 is the plane V

To investigate a relation between the multiplicity of solutions and source terms in a nonlinear suspension bridge equation

$$Lu + bu^+ = f \quad \text{in} \quad H \tag{1.6}$$

we consider the restrictions $\Phi|_{C_i} (1 \leq i \leq 4)$ of Φ to the cones C_i. Let $\Phi_i = \Phi|_{C_i}$, i.e.,

$$\Phi_i : C_i \to V.$$

For $i = 1, 3$, the image of Φ_i is R_i and $\Phi_i : C_i \to R_i$ is bijective. From now on, our goal is to find the image of C_i under Φ_i for $i = 2, 4$.

Suppose that γ is a simple path in C_2 without meeting the origin, and end points (initial and terminal) of γ lie on the boundary ray of C_2 and they are on each other boundary ray. Then the image of one end point of γ under Φ is on the ray $c_1(b+\lambda_{00})\phi_{00} + c_1(b+\lambda_{10})\phi_{10}, c_1 \geq 0$ (a boundary ray of R_1) and the image of the other end point of γ under Φ is on the ray $-c_1\lambda_{00}\phi_{00} + c_1\lambda_{10}, c_1 \geq 0$ (a boundary ray of R_3). Since

Φ is continuous, $\Phi(\gamma)$ is a path in V. By Lemma 1.2, $\Phi(\gamma)$ does not meet the origin. Hence the path $\Phi(\gamma)$ meets all rays (starting from the origin) in $R_1 \cup R_4'$ or all rays (starting from the origin) in $R_2' \cup R_3$.

Therefore it follows from Lemma 1.2 that the image $\Phi(C_2)$ of C_2 contains one of sets $R_1 \cup R_4'$ and $R_2' \cup R_3$.

Similarly, we have that the image $\Phi(C_4)$ of C_4 contains one of sets $R_1 \cup R_2'$ and $R_4' \cup R_3$.

LEMMA 1.3. *Let A be one of the sets $R_1 \cup R_4'$ and $R_2' \cup R_3$ such that it is contained in $\Phi(C_2)$. Let γ be any simple path in A with end points on ∂A, where each ray (starting from the origin) in A intersect only one point of γ. Then the inverse image $\Phi_2^{-1}(\gamma)$ of γ is a simple path in C_2 with end points on ∂C_2, where any ray (starting from the origin) in C_2 intersects only one point of this path.*

Proof. We note that $\Phi_2^{-1}(\gamma)$ is closed since Φ is continuous and γ is closed in V. Suppose that there is a ray (starting from the origin) in C_2 which intersects two points of $\Phi_2^{-1}(\gamma)$, say, p, $\alpha p\,(\alpha > 1)$. Then by Lemma 1.2,

$$\Phi_2(\alpha p) = \alpha \Phi_2(p),$$

which implies that $\Phi_2(p) \in \gamma$ and $\Phi_2(\alpha p) \in \gamma$. This contradicts that each ray (starting from the origin) in A intersect only one point of γ.

We regard a point p as a radius vector in the plane V. Then for a point p in V, we define the argument $\arg p$ of p by the angle from the positive ϕ_{00}-axis to p.

We claim that $\Phi_2^{-1}(\gamma)$ meets all ray (starting from the origin) in C_2. In fact, if not, $\Phi_2^{-1}(\gamma)$ is disconnected in C_2. Since $\Phi_2^{-1}(\gamma)$ is closed and meets at most one point of any ray in A, there are two points p_1 and p_2 in C_2 such that $\Phi_2^{-1}(\gamma)$ does not contain any point p with

$$\arg p_1 < \arg p < \arg p_2.$$

On the other hand, if we let l the segment with end points p_1 and p_2, then $\Phi_2(l)$ is a path in A, where $\Phi_2(p_1)$ and $\Phi_2(p_2)$ belong to γ. Choose a point q in $\Phi_2(l)$ that $\arg q$ is between $\arg \Phi_2(p_1)$ and $\arg \Phi_2(p_2)$. Then there exist a point q' such that $q' = \beta q$ for some $\beta > 0$. But $\Phi_2^{-1}(q')$ meets l and

$$\arg p_1 < \arg \Phi_2^{-1}(q') < \arg p_2,$$

which is a contradiction. This completes the lemma.

Similarly, we have the following lemma.

LEMMA 1.3'. *Let A be one of the sets $R_1 \cup R_2'$ and $R_4' \cup R_3$ such that it is contained in $\Phi(C_4)$. Let γ be any simple path in A with end points on ∂A, where each ray (starting from the origin) in A intersect only one point of γ. Then the inverse image $\Phi_4^{-1}(\gamma)$ of γ is a simple path in C_4 with end points on ∂C_4, where any ray (starting from the origin) in C_4 intersects only one point of this path.*

With Lemma 1.3 and Lemma 1.3', we have the following theorem, which is very important to investigate a relation between the multiplicity of solutions and source terms in a nonlinear suspension bridge equation.

THEOREM 1.2. *For $i = 2, 4$, if we let $\Phi_i(C_i) = R_i$, then R_2 is one of sets $R_1 \cup R_4'$, $R_2' \cup R_3$ and R_4 is one of sets $R_1 \cup R_2'$, $R_4' \cup R_3$.*

For each $1 \le i \le 4$, the restriction Φ_i maps C_i onto R_i. In particular, Φ_1 and Φ_3 are bijective.

If we determine the images $\Phi_i(C_i)$ for $i = 2, 4$, we can reveal a relation between the multiplicity of solutions and source terms in the nonlinear bridge equation. If the solution of (1.5) is in C_1, then it is positive. If the solution of (1.5) is in C_3, then it is negative. If the solution of (1.5) is in $\mathrm{Int} C_2 \cup C_4$, then it has both signs.

Therefore we can get the following.

REMARK. We conjecture that $\Phi_2(C_2) = R_1 \cup R_4'$, $\Phi_4(C_4) = R_1 \cup R_2'$. In this case we have: (i) If $f \in IntR_1$, then equation (1.1) has a positive solution and at least two sign changing solutions (ii) If $f \in IntR_2'$ or $f \in IntR_4'$, then equation (1.1) has at least one sign changing solution. (ii) If $f \in R_3$, then equation (1.1) has only the negative solution.

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