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ON THE COEFFICIENTS CHARACTERIZATION OF $BMOA$ FUNCTIONS ON THE UNIT BALL

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Let B denote the unit ball in $C^n (n \geq 1)$, and v the 2n-dimensional Lebesgue measure on B normalized so that $v(B) = 1$, while σ is the normalized surface measure on the boundary S of B.

For $z = (z_1, ..., z_n)$ and $w = (w_1, ..., w_n)$ in C^n, we let $< z, w >= z_1\overline{w_1} + \cdots + z_n\overline{w_n}$ so that $|z|^2 =< z, z >$. For $\alpha = (\alpha_1, \cdots, \alpha_n)$ with each α_i a nonnegative integer, we will write $|\alpha| = \alpha_1 + \cdots + \alpha_n, \alpha! = \alpha_1! \cdots \alpha_n!, z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \overline{w^\alpha} = \overline{w_1}^{\alpha_1} \cdots \overline{w_n}^{\alpha_n}$. The Hardy space $H^p (0 < p < \infty)$ is defined as that space of holomorphic functions f on B satisfying

$$(1) \qquad \|f\|_p^p = \sup_{0<r<1} \int_S |f(r\xi)|^p \, d\sigma(\xi) < \infty \quad .$$

The space $BMOA$ consists of the function $f \in H^1$ for which

$$(2) \qquad \|f\|_{BMO} = \sup \frac{1}{\sigma(Q)} \int_Q |f - f_Q| \, d\sigma < \infty \quad ,$$

where f_Q denotes the averages of f over Q and the supremum is taken over all $Q = Q_\delta(\xi) = \{\eta \in S, |1- < \eta, \xi > | < \delta\}$ for $\xi \in S$ and $0 < \delta \leq 2$. Here we have identified f with its boundary function. These spaces are discussed in more detail by Coifman, Rochberg, and Weiss [1].

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We use $H(B)$ to denote the class of all holomorphic functions on B. Every $f \in H(B)$ has a series expansion

$$f(z) = \sum_{\alpha \geq 0} a_\alpha z^\alpha \quad , \tag{3}$$

and the radial fractional derivative of order $\beta > 0$ of f is defined by

$$D^\beta f(z) = \sum_{\alpha \geq 0} (1 + |\alpha|)^\beta a_\alpha z^\alpha \quad .$$

A positive Borel measure μ on B is called a Carleson measure if there exists a constant C so that

$$\mu(B_\delta(\xi)) \leq C\delta^n$$

for every $\xi \in S$ and $\delta (0 < \delta \leq 2)$, where $B_\delta(\xi) = \{z \in B, |1- < z, \xi > | < \delta\}$ is said to be a Carleson region.

Here and elsewhere constants are denoted by C, which may indicate different constant from one occurrence to the next.

The fundamental properties of Carleson measure on the unit ball was obtained by Hörmander[4]. Our main result is the following theorem.

THEOREM. *Let* $f(z) = \sum_{\alpha \geq 0} a_\alpha z^\alpha \in H(B)$ *and* $\beta > 0$. *Then* $f \in BMOA$ *if and only if*

$$\sup_{w \in B} (1 - |w|^2)^n$$

$$\sum_{\alpha \geq 0} \frac{1}{(n + |\alpha|)^{2\beta}} \left| \sum_{|\gamma| \leq |\alpha|} \frac{(1 + |\alpha - \gamma|)^\beta (n + |\gamma| - 1)!}{\gamma!} a_{\alpha - \gamma} \overline{w}^\gamma \right|^2 < \infty.$$

Proof. In [5], Jevtić gave a characterization of $BMOA$ functions in terms of Carleson measure, that is , for $f \in H(B)$ and $\beta > 0$, $f \in BMOA$ if and only if $\left|D^\beta f(z)\right|^2 (1 - |z|^2)^{2\beta - 1} dv(z)$ is a Carleson measure on B. By lemma 4.1 in [2] we have that $f \in BMOA$ if and only if

$$\sup_{w \in B} \int_B \left|D^\beta f(z)\right|^2 (1 - |z|^2)^{2\beta - 1} \frac{(1 - |w|^2)^n}{|1- < z, w > |^{2n}} dv(z) < \infty \quad . \tag{4}$$

From [6] we have

$$
\begin{aligned}
& D^{\beta} f(z)(1-<z,w>)^{-n} \\
&= \sum_{\alpha\geq 0}(1+|\alpha|)^{\beta} a_{\alpha} z^{\alpha} \sum_{\alpha\geq 0} \frac{(n-1+|\alpha|)!}{(n-1)!\alpha!} \overline{w}^{\alpha} z^{\alpha} \\
&= \sum_{\alpha\geq 0} \sum_{|\gamma|\leq|\alpha|} \frac{(1+|\alpha-\gamma|)^{\beta}(n+|\gamma|-1)!}{(n-1)!\gamma!} a_{\alpha-\gamma} \overline{w}^{\gamma} z^{\alpha},
\end{aligned}
$$

by Parseval formula

$$
\int_S |\sum_{\alpha\geq 0} a_{\alpha}(r\xi)^{\alpha}|^2 d\sigma(\xi) = \sum_{\alpha\geq 0} |a_{\alpha}|^2 r^{2|\alpha|}
$$

we get

$$
\begin{aligned}
& \int_B \left|D^{\beta} f(z)\right|^2 (1-|z|^2)^{2\beta-1} \frac{(1-|w|^2)^n}{|1-<z,w>|^{2n}} dv(z) \\
&= 2n(1-|w|^2)^n \int_0^1 (1-r^2)^{2\beta-1} r^{2n-1} \times \\
&\times \int_S \left| \sum_{\alpha\geq 0} \sum_{|\gamma|\leq|\alpha|} \frac{(1+|\alpha-\gamma|)^{\beta}(n+|\gamma|-1)!}{(n-1)!\gamma!} a_{\alpha-\gamma} \overline{w}^{\gamma} (r\xi)^{\alpha} \right|^2 d\sigma(\xi) dr. \\
&= n(1-|w|^2)^n \sum_{\alpha\geq 0} \left| \sum_{|\gamma|\leq|\alpha|} \frac{(1+|\alpha-\gamma|)^{\beta}(n+|\gamma|-1)!}{(n-1)!\gamma!} a_{\alpha-\gamma} \overline{w}^{\gamma} \right|^2 \times \\
&\times \int_0^1 (1-r)^{2\beta-1} r^{|\alpha|+n-1} dr \\
&= n(1-|w|^2)^n \sum_{\alpha\geq 0} \left| \sum_{|\gamma|\leq|\alpha|} \frac{(1+|\alpha-\gamma|)^{\beta}(n+|\gamma|-1)!}{(n-1)!\gamma!} a_{\alpha-\gamma} \overline{w}^{\gamma} \right|^2 B(2\beta, |\alpha|+n),
\end{aligned}
$$

where $B(,)$ denotes the Beta function and we have

$$
B(2\beta, |\alpha|+n) = \frac{\Gamma(2\beta)\Gamma(|\alpha|+n)}{\Gamma(2\beta+|\alpha|+n)} \sim \frac{\Gamma(2\beta)}{(n+|\alpha|)^{2\beta}},
$$

where we used thus following results:

$$c_1(m+1)^{a-b} \le \frac{\Gamma(m+a)}{\Gamma(m+b)} \le c_2(m+1)^{a-b}$$

for $a, b > 0$ and positive integer m, where $c_j (j = 1, 2)$ is a positive constant which is independent of m. It is a trivial consequence of Stirling's formula.

So that (4) is equivalent to

$$\sup_{w\in B}(1-|w|^2)^n \sum_{\alpha\ge 0}\left|\sum_{|\gamma|\le|\alpha|}\frac{(1+|\alpha-\gamma|)^\beta(n+|\gamma|-1)!}{(n-1)!\gamma!}a_{\alpha-\gamma}\overline{w}^\gamma\right|^2\frac{1}{(n+|\alpha|)^{2\beta}}<\infty.$$

This finishes the proof.

By Jevtić's results [5] we can easily get that for $\beta > 0$ a function $f \in H(B)$ belongs to H^2 if and only if $\left|D^\beta f(z)\right|^2 (1-|z|^2)^{2\beta-1}dv(z)$ is a finite measure on B. From our theorem we have

COROLLARY. *Let* $f(z) = \sum_{\alpha\ge 0} a_\alpha z^\alpha \in H(B)$ *and* $\beta > 0$. *Then* $f \in H^2$ *if and only if*

$$\sum_{\alpha\ge 0}\left(\frac{1+|\alpha|}{n+|\alpha|}\right)^{2\beta}|a_\alpha|^2 < \infty \quad .$$

For the unit disk D in C^1, thus Theorem and Corollary with $\beta = 1$ show that

(a) $f(z) = \sum_{n=0}^\infty a_n z^n \in BMOA$ if and only if

$$\sup_{w\in D}(1-|w|^2)\sum_{n=0}^\infty \frac{1}{(1+n)^2}\left|\sum_{k=0}^n (1+n-k)a_{n-k}\overline{w}^k\right|^2 < \infty \quad .$$

(b) $f(z) = \sum_{n=0}^\infty a_n z^n \in H^2$ if and only if

$$\sum_{n=0}^\infty |a_n|^2 < \infty \quad .$$

REMARK. (b) is well-known result (see [3]). (a) has been probably obtained elsewhere.

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