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ON THE GROWTH OF ENTIRE FUNCTIONS SATISFYING SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction and statements of results

Let $f(z)$ be an entire function. Then the order $\rho(f)$ of f is defined by

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log^+ T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic of f (see [4]), $M(r,f) = \max_{|z|=r} |f(z)|$ and $\log^+ t = \max(\log t, 0)$.

The purpose of this note is to study on the growth of the solutions $f \not\equiv 0$ of the second order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

where $A(z)$ and $B(z) \not\equiv 0$ are entire functions.

It is well known that all solutions of (1.1) are entire functions, and that at least one of any two linearly independent solutions of (1.1) has infinite order if $A(z)$ is transcendental [6, pp.167-68]. It is also known that (i) if either $\rho(A) < \rho(B)$, or $A(z)$ is a polynomial and $B(z)$ is transcendental, or (ii) if either $\rho(B) < \rho(A) \le 1/2$, or $A(z)$ is transcendental with $\rho(A) = 0$ and $B(z)$ is a polynomial, then every solution $f \not\equiv 0$ of (1.1) is of infinite order [2,5,7].

Here we give a more precise estimation of the growth of the solutions of infinite order of (1.1) if $\rho(A) < \rho(B)$ or $\rho(B) < \rho(A) < 1/2$.

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THEOREM 1. *Let $A(z)$ and $B(z)$ be entire functions such that $\rho(A) < \rho(B)$ or $\rho(B) < \rho(A) < 1/2$. Then every solution $f \not\equiv 0$ of (1.1) satisfies*

$$\overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ T(r,f)}{\log r} \geq \max\{\rho(A), \rho(B)\}. \tag{1.2}$$

If $\rho(A)$ is a positive integer and $\rho(B) < \rho(A)$, (1.1) may have nonconstant solutions of finite order with $\rho(f) = \rho(A)$ [2, Examples 1 and 2]. Here we estimate the lower bound for the order of finite order solutions of (1.1).

THEOREM 2. *Let $A(z)$ and $B(z)$ be entire functions satisfying $\rho(B) < \rho(A)$. Then every solution $f \not\equiv 0$ of finite order of (1.1) satisfies $\rho(f) \geq \rho(A)$.*

REMARK. If $\rho(A) = \rho(B)$, the conclusion of Theorem 2 is in general false. Indeed, $f(z) = z$ solves $f'' + ze^z f' - e^z f = 0$.

If $\rho(A)$ is not a positive integer with $\rho(A) > 1/2$ and $\rho(B) < \rho(A)$, the possibility of solutions of finite order of (1.1) remains open. But G. Gunderson proved that if the growth of $B(z)$ dominates the growth of $A(z)$ in certain angular sector, then every solution $f \not\equiv 0$ of (1.1) (with $\rho(B) < \rho(A)$) has infinite order.

THEOREM A[2]. *Let $A(z)$ and $B(z)$ be entire functions such that for real constants α, β, θ_1, θ_2, where $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$, we have*

$$|A(z)| \leq \exp\{(o(1))|z|^\beta\} \tag{1.3}$$

and

$$|B(z)| \geq \exp\{(1 + o(1))\alpha|z|^\beta\} \tag{1.4}$$

as $z \to \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Then every solution $f \not\equiv 0$ of (1.1) has infinte order.

For $E \subset [0, \infty)$, the upper and the lower densities of E are defined by

$$\overline{dens} E = \overline{\lim_{r\to\infty}} \frac{m(E \cap [0,r])}{r}$$

and

$$\underline{dens}E = \lim_{r\to\infty} \frac{m(E\cap[0,r])}{r}$$

where $m(F)$ is the linear measure of a set F.

We generalize and improve Theorem A in the following theorem, in which the angular sector $\theta_1 \le \arg z \le \theta_2$ is replaced by a smaller set E.

THEOREM 3. *Let E be a set of complex numbers satisfying $\overline{dens}\{|z| : z\in E\} > 0$, and let $A(z)$ and $B(z)$ be entire functions which satisfy (1.3) and (1.4) respectively as $z\to\infty$ in E. Then every solution $f \not\equiv 0$ of (1.1) is of infinite order with*

$$\overline{\lim_{r\to\infty}} \frac{\log^+\log^+ T(r,f)}{\log r} \ge \beta. \tag{1.5}$$

REMARK. Let $A(z) = e^{-z}$ and $B(z) = -(e^{2z}+e^z+1)$, then $A(z)$ and $B(z)$ satisfy (1.3) and (1.4) with $\alpha = 2$ and $\beta = 1$ respectively on the positve real axis, and (1.1) has the solution $f(z) = \exp(e^z)$. Hence the inequality of (1.5) is sharp.

If $0 < \rho(B) < 1/2$, Theorem 3 is modified as the following theorem.

THEOREM 4. *Let $A(z)$ and $B(z)$ be entire functions where $0 < \rho(B) < 1/2$, and let there exist a real constant $\rho < \rho(B)$ and a set $E_\rho \subset [0,\infty)$ with $\underline{dens}E_\rho = 1$ such that for all $r\in E_\rho$ we have*

$$\min_{|z|=r} |A(z)| \le \exp(r^\rho). \tag{1.6}$$

Then every solution $f\not\equiv 0$ of (1.1) is of infinite order with

$$\overline{\lim_{r\to\infty}} \frac{\log^+\log^+ T(r,f)}{\log r} \ge \rho(B). \tag{1.7}$$

REMARK. Let $P(z)$ be a nonconstant polynomial and let $h(z)$ be an entire function satisfying $\rho(h) < \deg(P)$. Let $B(z)$ be an entire function with $0 < \rho(B) < 1/2$. Then, by Theorem 4, every solution $f \not\equiv 0$ of

$$f'' + he^P f' + Bf = 0 \tag{1.8}$$

is of infinite order with (1.7) since

$$\min_{|z|=r} |h(z)e^{P(z)}| \to 0$$

as $r \to \infty$. Actually, if $B(z)$ is a transcendental entire function with $\rho(B) \neq \deg(P)$, then every solution $f \not\equiv 0$ of (1.8) has infinite order [2, p.419].

2. Proofs of theorems

Our proofs depend mainly on the following lemmas.

LEMMA 1 [3]. *Let $f(z)$ be a nontrivial entire function, and let $\alpha > 1$ and $\epsilon > 0$ be given constants. Then there exist a constant $c > 0$ and a set $E_1 \subset [0, \infty)$ having finite linear measure such that for all z satisfying $|z| = r \notin E_1$ we have*

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq c[T(\alpha r, f)r^\epsilon \log T(\alpha r, f)]^k, \quad k \in \mathbf{N}.$$

LEMMA 2 [1]. *Let $f(z)$ be an entire function of order ρ where $0 < \rho < 1/2$, and let $\epsilon > 0$ be a given constant. Then there exists a set $E_2 \subset [0, \infty)$ with $\overline{dens} E_2 \geq 1 - 2\rho$ such that for all z satisfying $|z| = r \in E_2$, we have*

$$|f(z)| \geq \exp(r^{\rho - \epsilon}).$$

In addition we need the following elementary lemmas.

LEMMA 3. *Let $f(z)$ be a nonconstant entire function. Then there exists a real number R such that for all $r \geq R$ there exists z_r with $|z_r| = r$ satisfying*

$$\left|\frac{f(z_r)}{f'(z_r)}\right| \leq r.$$

Proof. We assume first that f has a zero inside the circle $|z| = R_1$ for some real number R_1. Suppose that $r \geq R_1$ and that f has no zero on the circle $|z| = r$. Then from the integration of f'/f around the circle $|z| = r$, it follows from the residue theorem that there exists a point z_r such that $|z_r| = r$ and $|f'(z_r)|/|f(z_r)| \geq 1/r$.

Next we assume that f has no zero in the complex plane. Then $f = e^g$ for some nonconstant entire function $g(z)$. Hence $f'/f = g'$ is a nonzero entire function. If g' is a constant, the result follows immediately. Otherwise, there is a real number R_2 such that $M(r, g') \geq 1$ for all $r \geq R_2$. Hence for all $r \geq R_2$ there exists a point z_r such that $|z_r| = r$ and $|f'(z_r)/f(z_r)| \geq 1$.

Thus the proof of Lemma 3 is complete.

LEMMA 4. *Let $f(z)$ be a nonconstant entire function of finite order. Then for any $\epsilon > 0$, there exists a set $E \subset [0, \infty)$ with $\overline{dens} E = 1$ such that*

$$M(r, f) \geq \exp(r^{\rho(f)-\epsilon})$$

for all $r \in E$.

Proof. Let $\alpha = \rho(f) - \epsilon$ and let $\beta = \rho(f) - \epsilon/2$. Then there is a sequence $\{r_n\}$ of real numbers for which we have $(r_n)^{\epsilon/2} \geq n^\alpha$ and

$$\log^+ M(r_n, f) \geq (r_n)^\beta.$$

Therefore

$$\log^+ M(r_n, f) \geq (n r_n)^\alpha$$

for all $n \in \mathbf{N}$. Setting $E = \cup_{n=1}^{\infty} [r_n, n r_n]$, we have $\overline{dens} E = 1$ and

$$\log^+ M(r, f) \geq r^\alpha$$

for all $r \in E$, since $M(r, f)$ is increasing. Hence the proof of Lemma 4 is complete.

Proof of Theorem 1. We first suppose that $\rho(A) < \rho(B)$, and that α and β are real numbers satifying $\rho(A) < \alpha < \beta < \rho(B)$. If $f(\not\equiv 0)$ is a solution of (1.1), then it follows that

$$|B| \leq \left|\frac{f''}{f}\right| + |A|\left|\frac{f'}{f}\right|. \tag{2.1}$$

By Lemma 1, there is a set $E_1 \subset [0, \infty)$ with a finite linear measure such that for all z satisfying $|z| = r \notin E_1$ we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq r[T(2r, f)]^3, k = 1, 2. \tag{2.2}$$

Also, by Lemma 4, there is a set $E_2 \subset [0, \infty)$ with $\overline{dens}E_2 = 1$ such that for all $r \in E_2$ we have

$$\exp(r^\beta) \leq M(r, B). \tag{2.3}$$

Hence by (2.1), (2.2) and (2.3), we get a set $E \subset [0, \infty)$ with $\overline{dens}E = 1$ such that for all $r \in E$ we have

$$\exp(r^\beta) \leq 2r\exp(r^\alpha)[T(2r, f)]^3.$$

Thus for all $r \in E$ we have

$$\exp\{(1 - o(1))r^\beta\} \leq [T(2r, f)]^3$$

as $r \to \infty$. Therefore

$$\overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ T(r, f)}{\log r} \geq \beta.$$

Since β is arbitrary, we get the result of the theorem.

Now suppose that $\rho(B) < \rho(A) < 1/2$, and let α and β are real numbers satisfying $\rho(B) < \beta < \alpha < \rho(A)$. If $f(\not\equiv 0)$ is a solution of (1.1), it follows that

$$|A| \leq \left|\frac{f''}{f'}\right| + |B|\left|\frac{f}{f'}\right|. \tag{2.4}$$

By Lemma 1, there is a set $E_3 \subset [0, \infty)$ with a finite linear measure such that for all z satisfying $|z| = r \notin E_3$ we have

$$\left|\frac{f''(z)}{f'(z)}\right| \leq r[T(2r, f')]^2. \tag{2.5}$$

Also, by Lemma 2, there is a set $E_4 \subset [0, \infty)$ with $\overline{dens} E_4 > 0$ such that for all z satisfying $|z| = r \in E_4$ we have

$$\exp(r^\alpha) \leq |A(z)|. \tag{2.6}$$

Hence by (2.4), (2.5) and (2.6), we get a set $E_5 \subset [0, \infty)$ with $\overline{dens} E_5 > 0$ such that for all z satisfying $|z| = r \in E_5$ we have

$$\exp(r^\alpha) \leq r[T(2r, f')]^2 + \exp(r^\beta)\left|\frac{f(z)}{f'(z)}\right|. \tag{2.7}$$

Now, by Lemma 3, there exists a number $R > 0$ such that for all $r \geq R$ there exists a z_r with $|z_r| = r$ satisfying

$$\left|\frac{f(z_r)}{f'(z_r)}\right| \leq r. \tag{2.8}$$

Therefore by (2.7) and (2.8) we get a set $E \subset [0, \infty)$ with $\overline{dens} E > 0$ such that for all $r \in E$ we have

$$\begin{aligned}\exp(r^\alpha) &\leq r[T(2r, f')]^2 + r\exp(r^\beta)\\ &\leq 2r\exp(r^\beta)[T(2r, f')]^2.\end{aligned}$$

Thus

$$\overline{\lim_{r\to\infty}}\frac{\log^+\log^+ T(r,f')}{\log r}\geq \alpha.$$

Since α is arbitrary, the result of the theorem follows from the fact that $T(r,f')\sim T(r,f)$ as $r\to\infty$ (see [4, p.58]).

The proof of Theorem 1 is now complete.

Proof of Theorem 2. Suppose that $f\not\equiv 0$ is a solution of (1.1) with $\rho(f)<\infty$. It follows from (1.1) that

$$-A=\frac{f''}{f'}+B\frac{f}{f'}.$$

Hence from Nevanlinna's fundamental results of meromorphic functions[4], we have

$$m(r,A)\leq m(r,B)+m(r,\frac{f}{f'})+O(\log r) \tag{2.9}$$

as $r\to\infty$. Here the notation $m(r,h)$ for a meromorphic function h is defined by

$$m(r,h)=\frac{1}{2\pi}\int_0^{2\pi}\log^+|h(re^{i\theta})|d\theta,$$

which is equal to $T(r,h)$ if h is entire. It follows from (2.9) that

$$T(r,A)-T(r,B)-O(\log r)\leq 2T(r,f)$$

as $r\to\infty$, since $m(r,\frac{f}{f'})\leq 2T(r,f)+O(1)$ as $r\to\infty$. Hence the result of the theorem follows from the fact that $\rho(B)<\rho(A)$.

Proof of Theorem 3. Suppose that $f\not\equiv 0$ is a solution of (1.1). Then from (2.1) and (2.2) there is a set $E_1\subset[0,\infty)$ with a finite linear measure such that for all z satisfying $|z|\notin E_1$ we have

$$|B(z)|\leq |z|[T(2|z|,f)]^3\{1+|A(z)|\}. \tag{2.10}$$

Also, by the hypothsis of the theorem, there exists a set E_2 with $\overline{dens}\{|z|:z\in E_2\}>0$ such that for all z satisfying $z\in E_2$ we have

$$|A(z)|\leq\exp\{(o(1))|z|^{\beta}\},$$

(2.11)

$$|B(z)| \geq \exp\{(1+o(1))\alpha|z|^{\beta}\}$$

as $z \to \infty$. Hence it follows from (2.10), (2.11) that for all z satisfying $z \in E_2$ and $|z| \notin E_1$ we have

$$\exp\{(1+o(1))\alpha|z|^{\beta}\} \leq |z|[1+\exp\{o(1)|z|^{\beta}\}][T(2|z|, f)]^3$$

as $z \to \infty$. Thus there exists a set $E \subset [0, \infty)$ with a positive upper density such that

$$\exp\{(1+o(1))\alpha|z|^{\beta}\} \leq [T(2r, f)]^3$$

as $r \to \infty$ in E. Therefore

$$\overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ T(r, f)}{\log r} \geq \beta.$$

This proves the theorem.

Proof of Theorem 4. Let $\rho < \rho(B)$ and let $f \not\equiv 0$ be a solution of (1.1). Suppose that $\rho < \beta < \rho(B)$ and that there is a set $E_\rho \subset [0, \infty)$ of lower density 1 satisfying (1.6). Set

$$E_1 = \{z : |z| = r \in E_\rho \text{ and } |A(z)| = \min_{|z|=r} |A(z)|\}$$

Then $\underline{dens}\{|z| : z \in E_1\} = 1$ and

$$|A(z)| \leq \exp(r^{\rho})$$

for all $z \in E_1$. Also, from Lemma 2, there is a set $E_2 \subset [0, \infty)$ of positive upper density such that for all z satisfying $|z| \in E_2$ we have

$$|B(z)| \geq \exp(r^{\beta}).$$

Now let $E = \{z \in E_1 : |z| \in E_2\}$. Then, with the set E and the number β, $A(z)$ and $B(z)$ satisfy the hypothesis of Theorem 3 respectively. Hence we conclude by Theorem 3 that

$$\overline{\lim_{r\to\infty}} \frac{\log^+ \log^+ T(r, f)}{\log r} \geq \beta.$$

Thus the result of the theorem follows since β is arbitrary.

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