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EIGEN 1–FORMS OF THE LAPLACIAN AND RIEMANNIAN SUBMERSIONS

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Let $\pi : Z \to Y$ be a fiber bundle where Y and Z are compact Riemannian manifolds without boundary. We are primarily interested in the case where π is a Riemannian submersion with minimal fibers; this is the case, for example, where Z is the sphere bundle of some vector bundle over Y or where Z is a principal bundle over Y.

Let $E(\lambda, \Delta_p^Y) \subset C^\infty \Lambda^p Y$ and $E(\lambda, \Delta_p^Z) \subset C^\infty \Lambda^p Z$ be the eigenspaces of the p form valued Laplacians on Y and on Z. We are interested in when eigenvalues can change; i.e. when there exists $\Phi \in E(\lambda, \Delta_p^Y)$ so $\pi^*\Phi \in E(\mu, \Delta_p^Z)$ with $\lambda \neq \mu$. The following summarizes briefly what is known about this question. Let ext and int be exterior and interior multiplication. Let θ be the (non-normalized) mean curvature vector. Let e_i and e^i be a local orthonormal frame field for the vertical distribution and co-distribution. Let f_a and f^a be a local orthonormal frame field for the horizontal distribution and co-distribution.
Let $\omega_{iab} := g(e_i, [f_a, f_b])$ be the curvature and let $\Omega := \Sigma_{a<b}\omega_{iab}$ $\text{ext}(e^i)\text{int}(f^a)\text{int}(f^b)$.

THEOREM 1. *Let $\pi : Z \to Y$ be a Riemannian submersion of closed manifolds.*

(a) *We have* $\Delta_p^Z \pi^* - \pi^* \Delta_p^Y = d^Z(\text{int}^Z(\theta) + \Omega)\pi^* + (\text{int}^Z(\theta) + \Omega)d^Z\pi^*$.

(b) *Let $p = 0$. Then the following conditions are equivalent:*

i) *We have* $\Delta_0^Z \pi^* = \pi^* \Delta_0^Y$.

ii) *For all* $\lambda \in \mathbb{R}$, $\exists \mu(\lambda) \in \mathbb{R}$ *so* $\pi^* E(\lambda, \Delta_0^Y) \subset E(\mu(\lambda), \Delta_0^Z)$.

iii) *The fibers are minimal.*

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(c) *Let* $1 \leq p \leq \dim(Y)$. *Then the following conditions are equivalent:*

i) *We have* $\Delta_p^Z \pi^* = \pi^* \Delta_p^Y$.

ii) *For all* $\lambda \in \mathbb{R}$, $\exists \mu(\lambda) \in \mathbb{R}$ *so* $\pi^* E(\lambda, \Delta_p^Y) \subset E(\mu(\lambda), \Delta_p^Z)$.

iii) *The fibers are minimal and the horizontal distribution is integrable.*

(d) *If* $0 \neq \Phi \in E(\lambda, \Delta_0^Y)$ *and* $\pi^*\Phi \in E(\mu, \Delta_0^Z)$, *then* $\lambda = \mu$

(e) *Let* $0 \leq \lambda < \mu$ *be given. If* $p \geq 2$, *there exists a Riemannian submersion* $\pi : Z \to Y$ *and* $\Phi_p \in E(\lambda, \Delta_p^Y)$ *so* $\pi^*\Phi_p \in E(\mu, \Delta_p^Z)$.

Assertion (a) is the fundamental formula in this subject; we refer to Watson [8] for the case $p = 0$ and to Goldberg and Ishihara [4] for the case $p > 0$. The equivalence of b-i and b-iii was proved by Watson and the equivalence of c-i and c-iii was proved by Goldberg and Ishihara; we refer to [3] for the equivalence of b-i and b-ii as well as c-i and c-ii. The observation that eigenvalues can not change if $p = 0$ is an immediate consequence of the maximum principal and assertion (a). Suppose that $p = 2$. Let $S^1 \to S^3 \to S^2$ be the Hopf fibration. The volume form ν_2 is harmonic on S^2 and $\pi^*\nu_2$ is not harmonic on S^3. We refer to Muto [5, 6] for a discussion of this and related examples; we also refer to [2] for other examples. Assertion (e) now follows by taking suitable products.

The case $p = 1$ has not been settled; it is not known if eigenvalues can change for 1-forms and this forms the focus of the present paper. We first show:

THEOREM 2. *Let* $\pi : Z \to Y$ *be a Riemannian submersion with minimal fibers. If* $0 \neq \Phi \in E(\lambda, \Delta_1^Y)$ *and* $\pi^*\Phi \in E(\mu, \Delta_1^Z)$, *then* $\lambda = \mu$.

Proof. We use Theorem 1 (a). Since the mean curvature vector vanishes and since Φ is a 1-form, we have $(\mu - \lambda)\pi^*\Phi = \omega_{iab}\,\mathrm{ext}(e^i)\,\mathrm{int}(f^a)\,\mathrm{int}(f^b) d\pi^*\Phi$. Since $\pi^*\Phi$ is a horizontal co-vector and $\mathrm{ext}(e^i)$ is a vertical co-vector, both sides of this equation vanish so $\lambda = \mu$. □

The eigenvalue $\lambda = 0$ is of special significance as we can use the DeRham theorem and the Hodge decomposition theorm to identify the kernel of the Laplacian with the cohomology groups of the manifold. This permits us to improve Theorem 2 for the zero mode spectrum by dropping the assumption that π is a Riemannian submersion with minimal fibers.

THEOREM 3. *Let $\pi : Z \to Y$ be a fiber bundle with connected fibers. If we have that $0 \neq \Phi \in E(0, \Delta_1^Y)$ and $\pi^*\Phi \in E(\mu, \Delta_1^Z)$, then $\mu = 0$.*

Proof. We argue by contradiction. Suppose $\mu \neq 0$. Let $\phi = \pi^*\Phi$. Since Φ is harmonic, it is both closed and co-closed. Thus $d^Z\phi = d^Z\pi^*\Phi = \pi^*d^Y\Phi = 0$ so that $\mu\phi = \Delta_1^Z\pi^*\Phi = d^Z\delta^Z\phi$ and so that $[\phi]$ is trivial in DeRham cohomology. This shows that π^* is not an injective map from $H^1(Y;\mathbb{R})$ to $H^1(Z;\mathbb{R})$.

Let M be a compact connected manifold. We use the universal coefficient theorem [7, Theorem 5.5.3 p243] to see that there is an equivalence of functors $H^1(M;\mathbb{R}) = \mathrm{Hom}(H_1(M);\mathbb{R})$. The Hurewicz theorem [7, Theorem 7.5.5 p398] shows that $H_1(M)$ is the Abelianization of $\pi_1(M)$. Consequently there is a natural identification of functors $H^1(M;\mathbb{R}) = \mathrm{Hom}(\pi_1(M);\mathbb{R})$. We use the long exact sequence of a fibration [7, Theorem 7.2.10 p377] and the fact that F is arc connected to see

$$\pi_1(F) \to \pi_1(Z) \to \pi_1(Y) \to 0$$

is part of a long exact sequence. This shows that push forward π_* is a surjective map from $\pi_1(Z)$ to $\pi_1(Y)$ and consequently pull back π^* is injective from $H^1(Y;\mathbb{R})$ to $H^1(Z;\mathbb{R})$. This provides the desired contradiction. □

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