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A BOUNDED CONVERGENCE THEOREM FOR THE OPERATOR–VALUED FEYNMAN INTEGRAL

BYUNG MOO AHN

1. Introduction

Fix $t > 0$. Denote by C^t the space of $\mathbb{R}$-valued continuous functions x on $[0,t]$. Let C_0^t be the Wiener space - $C_0^t = \{x \in C^t : x(0) = 0\}$ - equipped with Wiener measure m . Let F be a function from C^t to $\mathbb{C}$. Given $\lambda > 0, \psi \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$, let

$$(1.1) \qquad (K_\lambda(F)\psi)(\xi) = \int_{C_0^t} F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)\, dm(x).$$

DEFINITION. The operator- valued function space integral $K_\lambda(F)$ exists for $\lambda > 0$ if (1.1) defines $K_\lambda(F)$ as a bounded linear operator on $L^2(\mathbb{R})$. If, in addition, the operator-valued function $K_\lambda(F)$, as a function of λ, has an extension to an analytic function in $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : Re\lambda > 0\}$ and a strongly continuous function in $\tilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} : Re\lambda \geq 0, \lambda \neq 0\}$, we say that $K_\lambda(F)$ exists for $\lambda \in \tilde{\mathbb{C}}_+$. When λ is purely imaginary, $K_\lambda(F)$ is called the operator-valued Feynman integral of F.

For $s > 0, \lambda \in \tilde{\mathbb{C}}_+$ and $\psi \in L^2(\mathbb{R})$, let

$$(1.2) \qquad (exp[-s(H_0/\lambda)]\psi(\xi) = (\frac{\lambda}{2\pi s})^{\frac{1}{2}} \int_{\mathbb{R}} \psi(u) exp(-\frac{\lambda(u-\xi)^2}{2s})\, du.$$

The integral in (1.2) exists as an ordinary Lebesgue integral for $\lambda \in \mathbb{C}_+$, but, when λ is purely imaginary and ψ is not integrable, the integral

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should be interpreted in the mean as in the theory of the Fourier-Plancherel transform.

In this paper, θ is a bounded Borel measurable and everywhere defined real valued function on $(0,t)\times\mathbb{R}$ and we will let $M := ||\theta||_\infty$.

Let η be a finite signed Borel measure on $(0,t)$. Then η has a unique decomposition $\eta = \mu + \eta_d$ into a continuous part μ and a discrete part η_d[8]. The case where η_d has a finite support is most likely to be of interest. So, let

$$\eta_d = \sum_{j=1}^{N} \omega_j \delta_{\tau_j} \tag{1.3}$$

where δ_{τ_j} is as usual the Dirac measure at $\tau_j \in (0,t)$, $0 < \tau_1 < \cdots < \tau_N < t$ and $\omega_j \in \mathbb{R}$ for $j = 1, 2, \cdots, N$.

Let $\mathcal{M}(\mathbb{R})$ be the space of complex Borel measures on $\mathbb{R}$. The Fourier transform of $\nu \in \mathcal{M}(\mathbb{R})$ is the function $\hat{\nu}$ defined by

$$\hat{\nu}(u) = \int_{\mathbb{R}} e^{-iuv}\, d\nu(v), \qquad u \in \mathbb{R}. \tag{1.4}$$

Consider the functional

$$F(x) = \hat{\nu}\Big(\int_{(0,t)} \theta(s, x(s))\, d\eta(s)\Big), \qquad x \in C^t. \tag{1.5}$$

Then, by [1], $K_\lambda(F)$ exists for $\lambda > 0$. Also $K_\lambda(F)$ exists for $\lambda \in \tilde{\mathbb{C}}_+$ and is given by the generalized Dyson series, provided that

$$\int_{\mathbb{R}} e^{M||\eta|||u|}\, d|\nu|(u) < \infty, \tag{1.6}$$

i.e. for all $\lambda \in \tilde{\mathbb{C}}_+$, the following expansion of $K_\lambda(F)$ hold:

(1.7)

$$K_\lambda(F) = \sum_{n=0}^{\infty} n! a_n \sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N!}$$
$$\sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} L_0 L_1 \cdots L_N\, d\mu(s_1)\cdots d\mu(s_{q_0})$$

where $q_0, \cdots, q_N, k_1, \cdots, k_{N+1}$ are nonnegative integers,

(1.8)
$$\begin{aligned}\Delta_{q_0;k_1,\cdots,k_{N+1}} = \{(s_1, \cdots, s_{q_0}) \in (0,t)^{q_0} : 0 < s_1 < \cdots < s_{k_1} \\ < \tau_1 < s_{k_1+1} < \cdots < s_{k_1+k_2} < \tau_2 < s_{k_1+k_2+1} < \cdots \\ < s_{k_1+\cdots+k_N} < \tau_N < s_{k_1+\cdots k_N+1} < \cdots < s_{q_0} < t\}\end{aligned}$$

and, for $(s_1, \cdots, s_{q_0}) \in \Delta_{q_0;k_1,\cdots,k_{N+1}}$ *and* $r \in \{0, 1, \cdots, N\}$

(1.9)
$$\begin{aligned}L_r = [\theta(\tau_r)]^{q_r} e^{-(s_{k_1+\cdots+k_r+1}-\tau_r)(H_0/\lambda)}\theta(s_{k_1+\cdots+k_r+1}) \\ e^{-(s_{k_1+\cdots+k_r+2}-s_{k_1+\cdots+k_r+1})(H_0/\lambda)}\theta(s_{k_1+\cdots+k_r+2})\cdots \\ \theta(s_{k_1+\cdots+k_{r+1}})e^{-(\tau_{r+1}-s_{k_1+\cdots+k_{r+1}})(H_0/\lambda)}\end{aligned}$$

and

$$a_n = \frac{1}{n!}\int_{\mathbb{R}} (-i)^n u^n \, d\nu(u). \tag{1.10}$$

We use the conventions $\tau_0 = 0, \tau_{N+1} = t$ and $[\theta(\tau_0)]^{q_0} = 1$.

2. A stability theorem

We begin with a lemma which will be useful in the main theorems.

LEMMA. *Let $\{F_n(x)\}$ be a sequence of Borel measurable functionals such that $|F_n(x)| \leq B$ for some constant $B > 0$ and for all $n = 1, 2, 3, \cdots$. Further suppose that for every $\lambda > 0$*

$$F_n(\lambda^{-\frac{1}{2}}x + \xi) \to F(\lambda^{-\frac{1}{2}}x + \xi) \quad as \quad n \to \infty \tag{2.1}$$

for $m \times Leb. - a.e. (x, \xi)$. Then for every $\lambda > 0$

$$K_\lambda(F_n) \to K_\lambda(F) \quad strongly \quad as \quad n \to \infty.$$

Proof. Let $\lambda > 0$, $\psi \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$ be given. By (2.1), for $m \times Leb. - a.e.(x, \xi)$,

$$(2.2) \quad F_n(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi) \to F(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi).$$

Note that for every $x \in C_0^t$, for a.e. $\xi \in \mathbb{R}$ and for all $n = 1, 2, 3, \cdots$.

$$(2.3) \qquad |F_n(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)| \leq B|\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)|.$$

In view of (2.2), (2.3), and the Dominated Convergence Theorem for Wiener integrals,

$$(2.4) \qquad (K_\lambda(F_n)\psi)(\xi) \to (K_\lambda(F)\psi)(\xi) \; for \; Leb. - a.e. \, \xi.$$

Moreover, by (2.3) and Wiener's integration formula

(2.5)
$$\begin{aligned} |(K_\lambda(F_n)\psi)(\xi)| &\leq \int_{C_0^t} |F_n(\lambda^{-\frac{1}{2}}x + \xi)\psi(\lambda^{-\frac{1}{2}}x(t) + \xi)| \, dm(x) \\ &\leq B \int_{C_0^t} |\psi(\lambda^{-\frac{1}{2}}x + \xi)| \, dm(x) \\ &= B(e^{-t(H_0/\lambda)}|\psi|)(\xi) \end{aligned}$$

for every $n = 1, 2, \cdots$ and a.e. $\xi \in \mathbb{R}$.

Since $e^{-t(H_0/\lambda)}|\psi| \in L^2(\mathbb{R})$, using (2.4), (2.5) and the Lebesgue Dominated Convergence Theorem, we have

$$(2.6) \qquad K_\lambda(F_n) \to K_\lambda(F)$$

in $L^2(\mathbb{R})$.

The first theorem treats the case $\lambda > 0$.

THEOREM 1. *Let η be a finite signed Borel measure on $(0, t)$ and let $\nu \in \mathcal{M}(\mathbb{R})$. Suppose that θ and θ_m, $m = 1, 2, \cdots$ are all bounded by M on $(0, t) \times \mathbb{R}$. Let F be defined as (1.5) and F_m be defined as (1.5) except with θ replaced by θ_m. Assume that*

$$(2.7) \qquad \theta_m \to \theta$$

at each point of $(0,t)\times\mathbb{R}$ *as* $m\to\infty$. *Then for all* $\lambda>0$,

$$(2.8)\qquad K_\lambda(F_m)\to K_\lambda(F)\quad \text{strongly}\quad \text{as}\quad m\to\infty.$$

Proof. Let $\lambda>0, x\in C_0^t$ and $\xi\in\mathbb{R}$ be given. Since θ_m is bounded by M for all $m=1,2,\cdots$, by (2.7),

$$(2.9)\quad \int_{(0,t)}\theta_m(s,\lambda^{-\frac{1}{2}}x(s)+\xi)\,d\eta(s)\to\int_{(0,t)}\theta(s,\lambda^{-\frac{1}{2}}x(s)+\xi)\,d\eta(s)$$

Since $\hat{\nu}$ is continuous,

(2.10)

$$\hat{\nu}(\int_{(0,t)}\theta_m(s,\lambda^{-\frac{1}{2}}x(s)+\xi)\,d\eta(s))\to\hat{\nu}(\int_{(0,t)}\theta(s,\lambda^{-\frac{1}{2}}x(s)+\xi)\,d\eta(s))$$

i.e. $F_m(\lambda^{-\frac{1}{2}}x+\xi)\to F(\lambda^{-\frac{1}{2}}x+\xi)$. Note that

$$(2.11)\qquad |F_m(x)|=|\hat{\nu}(\int_{(0,t)}\theta_m(s,x(s))\,d\eta(s))| \le ||\hat{\nu}||$$

for all $x\in C^t$ and for all $m=1,2,\cdots$. Hence (2.11) and Lemma give the result for $\lambda>0$.

We now obtain a stability result for $\lambda\in\tilde{\mathbb{C}}_+$ under the assumption that the measure $|\nu|$ dies off rapidly at ∞.

THEOREM 2. *Let* θ *and* $\theta^{(m)}$, $m=1,2,\cdots$ *be everywhere defined* $\mathbb{R}$- *valued and Borel measurable functions bounded by* M *on all of* $(0,t)\times\mathbb{R}$. *Let* $\eta=\mu+\eta_d$ *be a finite signed Borel measure on* $(0,t)$ *where* η_d *is given by (1.3), and let* $\nu\in\mathcal{M}(\mathbb{R})$ *be such that*

$$(2.12)\qquad \int_{\mathbb{R}} e^{M||\eta|||u|}\,d|\nu|(u)<\infty.$$

Assume that

$$(2.13)\qquad \theta^{(m)}\to\theta\quad \text{as}\quad m\to\infty\quad \eta\times Leb.-a.e.\quad \text{on}\quad (0,t)\times\mathbb{R}.$$

Let F and $F^{(m)}$ be defined as in Theorem 1. Then for all $\lambda \in \tilde{\mathbb{C}}_+$

$$(2.14)\qquad K_\lambda(F^{(m)}) \to K_\lambda(F) \quad strongly \quad as \quad m \to \infty.$$

Further, the operator $K_\lambda(F)$ preserves the form of the operator $K_\lambda(F^{(m)})$; to be more specific,

(2.15)

$$K_\lambda(F^{(m)}) = \sum_{n=0}^{\infty} n!a_n \sum_{q_0+\cdot+q_N=n} \frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} L_0^{(m)}L_1^{(m)}\cdots L_N^{(m)}\, d\mu(s_1)\cdots d\mu(s_{q_0})$$

$$\longrightarrow$$

$$K_\lambda(F) = \sum_{n=0}^{\infty} n!a_n \sum_{q_0+\cdots+q_N=n} \frac{\omega^{q_1}\cdots\omega^{q_N}}{q_1!\cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} L_0L_1\cdots L_N d\mu(s_1)\cdots d\mu(s_{q_0})$$

strongly as $m \to \infty$;

where $q_0,\cdots,q_N,k_1,\cdots,\cdots,k_{N+1}$ are nonnegative integers and $\Delta_{q_0;k_1,\cdots,k_{N+1}}$ is given by (1.8) and, for $(s_1,\ldots,s_{q_0}) \in \Delta_{q_0;k_1,\cdots,k_{N+1}}$ and $r \in \{0,1,\cdots,N\}$, L_r is given by (1.9), and $L_r^{(m)}$ is given as in (1.9) except with θ replaced by $\theta^{(m)}$ and a_n is given by (1.10).

Proof. Let $\psi \in L^2(\mathbb{R})$ be given. Let $\theta^{(m)}(s)$ denote the operator of multiplication by $\theta^{(m)}(s,\cdot)$ so that $(\theta^{(m)}(s)\psi)(\xi) = \theta^{(m)}(s,\xi)\psi(\xi)$ for all $\xi \in \mathbb{R}$. So, by (2.13)

$$(2.16)\qquad (\theta^{(m)}(s)\psi)(\xi) \to (\theta(s)\psi)(\xi) \quad as \quad m \to \infty$$

Leb. $-$ *a.e.* for η $-$ *a.e.* $s \in (0,t)$. But

$$(2.17)\qquad \begin{aligned} &|(\theta^{(m)}(s)\psi)(\xi) - (\theta(s)\psi)(\xi)|^2 \\ &\le (|\theta^{(m)}(s,\xi)||\psi(\xi)| + |\theta(s,\xi)||\psi(\xi)|)^2 \\ &\le 4M^2|\psi(\xi)|^2. \end{aligned}$$

Since $\psi \in L^2(\mathbb{R})$, next using (2.16) and the Lebesgue Dominated Convergence Theorem, we have

$$(2.18) \qquad ||\theta^{(m)}(s)\psi - \theta(s)\psi||_2 \to 0 \quad as \quad m \to \infty;$$

$i.e.\, \theta^{(m)}(s) \to \theta(s)$ strongly as $m \to \infty$ for $\eta - a.e.\, s$.

Using (2.18) and the fact that the composition of operator is jointly continuous in the strong operator topology when the operators involved are uniformly bounded we see that

$$(2.19) \qquad L_0^{(m)} L_1^{(m)} \cdots L_N^{(m)} \to L_0 L_1 \cdots L_N$$

strongly $\mu \times \cdots \times \mu - a.e.$ in $\Delta_{q_0;k_1,\cdots,k_{N+1}}$.

Note that $L_0^{(m)} L_1^{(m)} \cdots L_N^{(m)}(\lambda; s_1, \cdots, s_{q_0})$ is strongly measurable [7].

Since $\theta^{(m)}$ is bounded by M and $||e^{-s(H_0/\lambda)}|| \leq 1$

$$(2.20) \qquad ||L_0^{(m)} L_1^{(m)} \cdots L_N^{(m)} \psi|| \leq M^n ||\psi||_2.$$

Further,

(2.21)

$$\begin{aligned} \sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} & ||L_0^{(m)} L_1^{(m)} \cdots L_N^{(m)} \psi||\, d|\mu|(s_1) \cdots d|\mu|(s_{q_0}) \\ & \leq M^n ||\psi||_2 \int_{\Delta_{q_0}} d|\mu|(s_1) \cdots d|\mu|(s_{q_0}) \\ & \leq M^n ||\psi||_2 \frac{||\mu||^{q_0}}{q_0!} < \infty. \end{aligned}$$

So, $L_0^{(m)} L_1^{(m)} \cdots L_N^{(m)}$ is Bochner integrable over $\Delta_{q_0;k_1,\cdots,k_{N+1}}$. Note that since μ is a finite signed Borel measure on $(0, t)$

$$(2.22) \qquad M^n ||\mu|| \in L_1(\Delta_{q_0;k_1,\cdots,k_{N+1}}, \mu \times \cdots \times \mu).$$

Therefore, using (2.19) and the Dominated Convergence Theorem for the Bochner integral [3], we have that $L_0 L_1 \cdots L_N$ is Bochner integrable and

$$(2.23)\qquad \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} L_0^{(m)} L_1^{(m)} \cdots L_N^{(m)}\, d\mu(s_1)\cdots d\mu(s_{q_0})$$

$$\longrightarrow \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} L_0 L_1 \cdots L_N\, d\mu(s_1)\cdots d\mu(s_{q_0})$$

in $L^2(\mathbb{R})$. Set

$$(2.24)\qquad \mathcal{L}_n^{(m)} := \sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} L_0^{(m)} L_1^{(m)} \cdots L_N^{(m)}\, d\mu(s_1)\cdots d\mu(s_{q_0})$$

and

$$(2.25)\qquad \mathcal{L}_n := \sum_{q_0+\cdots+q_N=n} \frac{\omega_1^{q_1}\cdots\omega_N^{q_N}}{q_1!\cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0} \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} L_0 L_1 \cdots L_N\, d\mu(s_1)\cdots d\mu(s_{q_0}).$$

Then

$$(2.26)\qquad \mathcal{L}_n^{(m)} \to \mathcal{L}_n \quad \textit{strongly} \quad \textit{as} \quad m \to \infty.$$

Furthermore,
(2.27)

$$\begin{aligned}
||\mathcal{L}_n^{(m)}\psi|| \le & \sum_{q_0+\cdots+q_N=n} \frac{|\omega_1|^{q_1}\cdots|\omega_N|^{q_N}}{q_1!\cdots q_N!} \sum_{k_1+\cdots+k_{N+1}=q_0} \\
& \int_{\Delta_{q_0;k_1,\cdots,k_{N+1}}} ||L_0^{(m)}L_1^{(m)}\cdots L_N^{(m)}\psi||\, d|\mu|(s_1)\cdots d|\mu|(s_{q_0}) \\
\le & \sum_{q_0+\cdots+q_N=n} \frac{|\omega_1|^{q_1}\cdots|\omega_N|^{q_N}}{q_1!\cdots q_N!} M^n||\psi|| \int_{\Delta_{q_0}} d|\mu|(s_1)\cdots d|\mu|(s_{q_0}) \\
= & \sum_{q_0+\cdots+q_N=n} \frac{|\omega_1|^{q_1}\cdots|\omega_N|^{q_N}}{q_1!\cdots q_N!} M^n||\psi|| \frac{||\mu||^{q_0}}{q_0!} \\
= & M^n||\psi||\frac{1}{n!}\sum_{q_0+\cdots+q_N=n}\frac{n!}{q_0!\cdots q_N!}||\mu||^{q_0}|\omega_1|^{q_1}\cdots|\omega_N|^{q_N} \\
= & M^n||\psi||\frac{(||\mu||+|\omega_1|+\cdots|\omega_N|)^n}{n!} \\
= & M^n\frac{||\eta||^n}{n!}||\psi||.
\end{aligned}$$

Similarly

(2.28) $$||\mathcal{L}_n\psi|| \le M^n\frac{||\eta||^n}{n!}||\psi||.$$

Let $\epsilon > 0$ be given. Using (2.12), take N_0 so large that

(2.29) $$\sum_{n=N_0+1}^{\infty} |a_n|M^n||\eta||^n||\psi|| < \frac{\epsilon}{4}.$$

Now using (2.26), let N be so large that for $m \ge N$

(2.30) $$\sum_{n=1}^{N_0} n!|a_n|||\mathcal{L}_n^{(m)}\psi - \mathcal{L}_n\psi|| < \frac{\epsilon}{2}.$$

Now, let $m \geq N$. Then using (2.30), (2.27), (2.28) and (2.29),
(2.31)

$$\begin{aligned}
&||K_\lambda(F^{(m)})\psi - K_\lambda(F)\psi|| \\
&= ||\sum_{n=0}^{\infty} n!a_n\mathcal{L}_n^{(m)}\psi - \sum_{n=0}^{\infty} n!a_n\mathcal{L}_n\psi|| \\
&= ||\sum_{n=0}^{N_0}(n!a_n\mathcal{L}_n^{(m)}\psi - n!a_n\mathcal{L}_n\psi) + \sum_{n=N_0+1}^{\infty}(n!a_n\mathcal{L}_n^{(m)}\psi - n!a_n\mathcal{L}_n\psi)|| \\
&\leq \sum_{n=0}^{N_0} n!|a_n|||\mathcal{L}_n^{(m)}\psi - \mathcal{L}_n\psi|| + \sum_{n=N_0+1}^{\infty} n!|a_n|||\mathcal{L}_n^{(m)}\psi|| \\
&+ \sum_{n=N_0+1}^{\infty} n!|a_n|||\mathcal{L}_n\psi|| \\
&< \frac{\epsilon}{2} + \sum_{n=N_0+1}^{\infty} n!|a_n|\frac{M^n||\eta||^n}{n!}||\psi|| + \sum_{n=N_0+1}^{\infty} n!|a_n|\frac{M^n||\eta||^n}{n!}||\psi|| \\
&= \frac{\epsilon}{2} + 2\sum_{n=N_0+1}^{\infty} |a_n|M^n||\eta||^n||\psi|| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad as \quad desired.
\end{aligned}$$

We can obtain a corollary immediately from a simple standard result of functional analysis.

COROLLARY 1. *Let the hypotheses of Theorem 2 be satisfied and suppose that* $||\psi_m - \psi|| \to 0$ *as* $m \to \infty$. *Then*

$$||K_\lambda(F^{(m)})\psi_m - K_\lambda(F)\psi|| \to 0 \qquad as \quad m \to \infty.$$

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DEPARTMENT OF MATHEMATICS, SOONCHUNHYANG UNIVERSITY, ASHAN 337-880, KOREA