Bull. Korean Math. Soc. **33** (1996), No. 3, pp. 455–463

CONFORMALLY INVARIANT TENSORS ON HERMITIAN MANIFOLDS

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In [3] and [4], Kitahara, Pak and the author obtained the conformally invariant tensor B_0, which is an algebraic Hermitian analogue of the Weyl conformal curvature tensor W in the Riemannian geometry, by the decomposition of the curvature tensor H of the Hermitian connection and the notion of semi-curvature-like tensors of Tanno (see [7]). In [5], the author defined a conformally invariant tensor $\mathcal{B}_0$ on a Hermitian manifold as a modification of B_0. Moreover he introduced the notion of local conformal Hermitian-flatness of Hermitian manifolds and proved that the vanishing of this tensor $\mathcal{B}_0$ together with some condition for the scalar curvatures is a necessary and sufficient condition for a Hermitian manifold to be locally conformally Hermitian-flat.

Recently, the author in [6] introduced the new conformally invariant tensor $\mathfrak{B}$ which is naturally obtained from the local conformal Hermitian-flatness of a Hermitian manifold and proved that, without any conditions for the scalar curvatures, the vanishing of this tensor $\mathfrak{B}$ is equivalent to local conformal Hermitian-flatness of the Hermitian manifold. We understand that the tensor $\mathfrak{B}$ is a geometric Hermitian analogue of the Weyl conformal curvature tensor W.

In this note, we shall discuss the relations among the conformally invariant tensors $\mathfrak{B}$, $\mathcal{B}_0$, and B which is derived from the curvature decomposition of the Hermitian connection. We would like to claim the usefulness of $\mathfrak{B}$ for local conformal Hermitian-flatness of Hermitian manifolds.

Throughout this note, we always assume the differentiability of class C^∞ and assume manifolds to be connected and without boundary. The complex dimensions of almost Hermitian manifolds are assumed to be

Received January 8, 1996.
1991 AMS Subject Classification: 53C55, 53C25.
Keywords: Hermitian connection, conformal invariant tensors.

no less than 2, and given a manifold M, $\mathfrak{X}(M)$ denotes the Lie algebra of all vector fields on M.

1. Preliminaries

Let (M, J, g) be a Hermitian manifold of complex dimension m. The Hermitian connection D of (M, J, g) is defined by

$$
\begin{aligned}
(1.1) \qquad 4g(D_X Y, Z) = {} & 2Xg(Y, Z) - 2JXg(JY, Z) \\
& + g(\mathcal{V}(X, Y), Z) - g(\mathcal{V}(X, Z), Y),
\end{aligned}
$$

where $X, Y, Z \in \mathfrak{X}(M)$ and $\mathcal{V}(X, Y) = [JX, JY] + [X, Y] - J[X, JY] + J[JX, Y]$ (cf. [6]). Then D has the following properties: $Dg = 0$, $DJ = 0$ and $T(JX, Y) = T(X, JY)$ for all $X, Y \in \mathfrak{X}(M)$, where T denotes the torsion tensor of D, i.e., $T(X, Y) = D_X Y - D_Y X - [X, Y]$. The curvature tensor H of D is defined by $H(X, Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$ for all $X, Y, Z \in \mathfrak{X}(M)$.

LEMMA 1.1. *H satisfies the following equations: For any $X, Y, Z, W \in \mathfrak{X}(M)$,*

$$H(X, Y, Z, W) = -H(Y, X, Z, W) = -H(X, Y, W, Z),$$

$$H(JX, JY, Z, W) = H(X, Y, JZ, JW) = H(X, Y, Z, W),$$

$$\mathfrak{S}_{X,Y,Z}\{H(X, Y)Z\} = \mathfrak{S}_{X,Y,Z}\{T(T(X, Y), Z) + (D_X T)(Y, Z)\},$$

$$\mathfrak{S}_{X,Y,Z}\{(D_X H)(Y, Z) + H(T(X, Y), Z)\} = 0,$$

where $H(X, Y, Z, W) = g(H(Z, W)Y, X)$ and $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y, and Z.

We define three tensors R, S, and Q which are analogous to the Ricci tensor in the Riemannian geometry: For any $X, Y \in \mathfrak{X}(M)$,

$$R(X, Y) = \sum_{\alpha=1}^{m} H(e_\alpha, Je_\alpha, X, JY),$$

$$S(X, Y) = \sum_{\alpha=1}^{m} H(X, JY, e_\alpha, Je_\alpha),$$

$$Q(X,Y)=\frac{1}{2}\sum_{\alpha=1}^{m}\{H(e_\alpha,X,e_\alpha,Y)+H(Je_\alpha,X,Je_\alpha,Y)$$
$$+H(e_\alpha,Y,e_\alpha,X)+H(Je_\alpha,Y,Je_\alpha,X)\},$$

where $\{e_1,...,e_m,Je_1,...,Je_m\}$ is a local adapted orthonormal frame field of (M,J,g). Then, by Lemma 1.1, these tensors are symmetric and compatible with J. Thus, we can associate 2-forms ρ_R,ρ_S, and ρ_Q with the Ricci-type tensors R,S, and Q respectively in the usual manner:

$$\rho_R(X,Y)=R(X,JY),$$
$$\rho_S(X,Y)=S(X,JY),$$
$$\rho_Q(X,Y)=Q(X,JY).$$

In particular, ρ_R is called the *Ricci form* of the Hermitian connection D and it is closed.

Moreover, we define two scalar curvatures s and $\hat{s}$ which are analogous to the scalar curvature in the Riemannian geometry:

$$s=2\sum_{\alpha=1}^{m}R(e_\alpha,e_\alpha)=2\sum_{\alpha=1}^{m}S(e_\alpha,e_\alpha),\quad \hat{s}=2\sum_{\alpha=1}^{m}Q(e_\alpha,e_\alpha).$$

2. Conformally Invariant Tensors

Let (M,J,g) be a Hermitian manifold of complex dimension m. Then, consider a conformal change $g'=e^{-\sigma}g$ of metric g where $\sigma\in C^\infty(M)$. Denoting by D',H' and $\rho_{R'}$ the Hermitian connection, the curvature tensor and the Ricci form with respect to g' respectively, we have

$$D'_XY=D_XY-\frac{1}{2}d\sigma(X)Y-\frac{1}{2}d^{\mathbf{c}}\sigma(X)JY, \tag{2.1}$$

$$H'=e^{-\sigma}(H-\Omega\otimes dd^{\mathbf{c}}\sigma),\qquad \rho_{R'}=\rho_R-mdd^{\mathbf{c}}\sigma, \tag{2.2}$$

where Ω denotes the fundamental form of (M, J, g), i.e., $\Omega(X, Y) = g(X, JY)$ for any $X, Y \in \mathfrak{X}(M)$ and d^c is a differential operator (see [1] for the definition). From (2.2), we have $H' - \frac{1}{m}\Omega' \otimes \rho_{R'} = e^{-\sigma}(H - \frac{1}{m}\Omega \otimes \rho_R)$. Thus we naturally obtain a tensor field $\mathfrak{B}$ defined by

$$\mathfrak{B} = H - \frac{1}{m}\Omega \otimes \rho_R. \tag{2.3}$$

Then we have

THEOREM 2.1. ([6]) *The tensor $\mathfrak{B}$ is conformally invariant.*

Next we recall the definition of the tensor $\mathcal{B}_0$ in [5] which is an algebraic Hermitian analogue of the Weyl conformal curvature tensor W in the Riemannian geometry. The complex structure J induces the splitting of the complexified tangent bundle $T_{\mathbf{C}}M \simeq TM \otimes_{\mathbf{R}} \mathbf{C}$ into two complementary subbundles, conjugate to each other:

$$T_{\mathbf{C}}M \simeq T^{1,0}M \oplus T^{0,1}M, \tag{2.4}$$

where, at each point x of M, the fiber $T_x^{1,0}M$ (resp. $T_x^{0,1}M$) is the eigenspace of J_x relative to eigenvalue $\sqrt{-1}$ (resp. $-\sqrt{-1}$). The elements of $T^{1,0}M$ (resp. $T^{0,1}M$) are the (complex) vectors of type (1,0) (resp. of type (0,1)).

The above splitting (2.4) of the complexified tangent bundle $T_{\mathbf{C}}M$ extends a splitting into types of the whole tensor bundle. In particular, we have

$$\bigwedge\nolimits_{\mathbf{C}}^{r} M = \sum_{p+q=r} \bigwedge\nolimits^{p}(T^{1,0}M)^* \otimes \bigwedge\nolimits^{q}(T^{0,1}M)^*,$$

where $\bigwedge_{\mathbf{C}}^{r} M$ is the bundle of the $\mathbf{C}$-valued r-forms and $\bigwedge^{p}(T^{1,0}M)^*$ (resp. $\bigwedge^{q}(T^{0,1}M)^*$) denotes the bundle of the $\mathbf{C}$-linear p-forms (resp. q-forms) on $T^{1,0}M$ (resp. $T^{0,1}M$). Sections of the bundle $\bigwedge_{\mathbf{C}}^{p,q} M = \bigwedge^{p}(T^{1,0}M)^* \otimes \bigwedge^{q}(T^{0,1}M)^*$ are complex forms of type (p, q). In particular, if $p = q$, we denote by $\bigwedge^{p,p} M$ the bundle of real forms of type (p, p).

By means of the fundamental form Ω, we define the linear operator L as $L\varphi = \Omega \wedge \varphi$ for any form φ and denote by Λ its formal adjoint operator. A form φ is said to be primitive if $\Lambda\varphi = 0$. Then, for any 2-form φ of type (1,1), we have the following Lefschetz decomposition:

$$\varphi = \frac{\Lambda\varphi}{m}\Omega + \varphi_0, \tag{2.5}$$

where φ_0 is the primitive part of φ.

In section 1, we saw that the curvature tensor H of the Hermitian connection satisfies the following relations: $H(X,Y) \circ J = J \circ H(X,Y)$, $H(JX,JY) = H(X,Y)$ for any $X, Y \in \mathfrak{X}(M)$. From these relations, we may understand the curvature tensor H as follows:

(A) H is a section of the vector bundle $\bigwedge^{1,1} M \otimes \bigwedge^{1,1} M$.

(B) H is an endmorphism of the vector bundle $\bigwedge^{1,1} M$.

From the viewpoint of (A), H is decomposed as follows (cf. [5]):

$$H = \frac{s}{2m^2}\Omega \otimes \Omega + \frac{1}{m}\Omega \otimes \rho_{R,0} + \frac{1}{m}\rho_{S,0} \otimes \Omega + B, \tag{2.6}$$

where $\rho_{R,0}$ (resp. $\rho_{S,0}$) is the primitive part of ρ_R (resp. ρ_S), i.e., $\rho_{R,0} = \rho_R - \frac{s}{2m}\Omega$ (resp. $\rho_{S,0} = \rho_S - \frac{s}{2m}\Omega$). Let $\bigwedge_0^{1,1} M$ be the vector bundle of real primitive 2-forms of type (1,1). Then $\bigwedge_0^{1,1} M$ is a subbundle of $\bigwedge^{1,1} M$ and the tensor B above is a section of the bundle $\bigwedge_0^{1,1} M \otimes \bigwedge_0^{1,1} M$. By the Lefschetz decomposition (2.5), the bundle $\bigwedge^{1,1} M$ decomposes as $\bigwedge^{1,1} M = \bigwedge_0^{1,1} M \otimes \mathbf{R} \cdot \Omega$, where $\mathbf{R} \cdot \Omega$ denotes the trivial (real) line subbundle of $\bigwedge^{1,1} M$ generated by Ω. Considered as an endomorphism of $\bigwedge^{1,1} M$, we have $B(\mathbf{R} \cdot \Omega) = 0$. Therefore the trace of B as an endomorphism of $\bigwedge^{1,1} M$ is equal to one as an endomorphism of $\bigwedge_0^{1,1} M$. Considered as an endomorphism of $\bigwedge_0^{1,1} M$, the tensor B decomposes as follows:

$$B = \frac{\operatorname{tr} B}{m^2 - 1} Id_{|\bigwedge_0^{1,1} M} + B_0, \quad \operatorname{tr} B = \frac{m\hat{s} - s}{2m}, \tag{2.7}$$

where B_0 is the trace-free part of B. Then the components of the tensor B_0 are expressed in terms of a complex local coordinate system $z^1, ..., z^m$ as follows:

$$B_{0,i\bar{j}k\bar{l}} = H_{i\bar{j}k\bar{l}} + \frac{1}{m}\left(g_{i\bar{j}}R_{k\bar{l}} + S_{i\bar{j}}g_{k\bar{l}}\right) - \frac{m\hat{s} + (m^2-2)s}{2m^2(m^2-1)}\, g_{i\bar{j}}g_{k\bar{l}} + \frac{m\hat{s} - s}{2m(m^2-1)}\, g_{i\bar{l}}g_{k\bar{j}}.$$

Then we have

THEOREM 2.2. ([5]) *The tensor B_0 is conformally invariant.*

In a Riemannian manifold (M, g), Tanno in [7] called a tensor K of type (0,4) a *semi-curvature-like tensor* if it satisfies the following conditions:

(1) $K(X,Y,Z,W) = -K(Y,X,Z,W) = -K(X,Y,W,Z)$,

(2) $K(X,Y,Z,W) + K(X,Z,W,Y) + K(X,W,Y,Z) = 0$.

Moreover, if a tensor K is expressed as a sum of tensor each of which contains just one of the curvature tensor of Levi-Civita connection, the Ricci tensor and the scalar curvature, then he called that K is of *curvature degree* 1. By means of the notion of semi-curvature-like tensors of curvature degree 1, he obtained algebraic characterizations of the Weyl's conformal curvature tensor in the Riemannian case and the Bochner curvature tensor in the Kählerian case.

Now we consider a tensor K of type (0,4) defined on any Hermitian manifold (M, J, g) which satisfies the following conditions:

(i) $K(X,Y,Z,W) = -K(Y,X,Z,W) = -K(X,Y,W,Z)$,

(ii) if (M, J, g) is Kählerian, then $K(X,Y,Z,W) + K(X,Z,W,Y) + K(X,W,Y,Z) = 0$.

The curvature tensor H of the Hermitian connection satisfies (i) and (ii). We shall call such a tensor K a *Hermitian semi-curvature-like tensor*. For simplicity, put

$$R_1(X,Y) = R(X,Y), \quad R_2(X,Y) = S(X,Y), \quad R_3(X,Y) = Q(X,Y),$$
$$s_1 = s, \quad s_2 = \hat{s}.$$

Then we have

LEMMA 2.1. *In a Hermitian manifold (M, J, g), every Hermitian semi-curvature-like tensor K of curvature degree 1 which is constructed by $(H, R, S, Q, s, \hat{s}, g, J)$ is of the form:*

$$K = pH + \sum_{a=1}^{3} (q_a\, g \triangle R_a + r_a\, \Omega \triangle \rho_a + u_a\, \Omega \otimes \rho_a + v_a\, \rho_a \otimes \Omega)$$
$$+ \sum_{b=1}^{2} s_b \{ w_b (\Omega \triangle \Omega + 4\, \Omega \otimes \Omega) + x_b\, g \triangle g \}$$

with $\sum_{a=1}^{3} u_a = \sum_{a=1}^{3} v_a = 2 \sum_{a=1}^{3} r_a$, *where* $p, q_a, r_a, u_a, v_a, w_b, x_b \in \mathbf{R}$, $\rho_a(X, Y) = R_a(X, JY)$ *and* $A \triangle B$ *denotes the product of two tensors* A, B *of type* $(0, 2)$ *as follows:*

$$(A \triangle B)(X, Y, Z, W) = A(X, Z) B(Y, W) - A(X, W) B(Y, Z)$$
$$+ B(X, Z) A(Y, W) - B(X, W) A(Y, Z).$$

If a tensor $\mathcal{B}_0$ as in Lemma 2.1 has the components such that $\mathcal{B}_0(Z_i, Z_{\bar{j}}, Z_k, Z_{\bar{l}}) = B_{0, i\bar{j}k\bar{l}}$ and if it is conformally invariant, then we obtain

$$p = 1, \quad q_1 = -\frac{1}{4m(m-1)}, \quad q_2 = \frac{1}{4m}, \quad q_3 = \frac{1}{4(m-1)},$$
$$u_1 = -\frac{1}{m}, \quad v_2 = -\frac{1}{m}, \quad u_2 = u_3 = v_1 = v_3 = 0,$$
$$w_1 = \frac{m^2 - 2}{8m^2(m^2 - 1)}, \quad w_2 = \frac{m}{8m^2(m^2 - 1)},$$
$$x_1 = -\frac{m^2 - 2m - 2}{8m^2(m^2 - 1)}, \quad x_2 = -\frac{2m + 1}{8m(m^2 - 1)}.$$

Therefore we obtain

$$\begin{aligned} \mathcal{B}_0 = {} & H + \frac{1}{2m}(g \triangle P - \Omega \triangle \rho_P - 2\, \Omega \otimes \rho_R - 2\, \rho_S \otimes \Omega) \\ & + \frac{m\hat{s} + (m^2 - 2)s}{8m^2(m^2 - 1)} (\Omega \triangle \Omega + 4\, \Omega \otimes \Omega) \\ & - \frac{m(2m+1)\hat{s} + (m^2 - 2m - 2)s}{8m^2(m^2 - 1)}\, g \triangle g, \end{aligned} \tag{2.8}$$

where $P = \dfrac{1}{2(m-1)}(mQ - R) + \dfrac{1}{2}S$ and ρ_P denotes the 2-form associated to P. Then we have

THEOREM 2.3. ([5]) *The tensor $\mathcal{B}_0$ above is conformally invariant.*

Now we shall recall the definitions of Hermitian-flatness and local conformal Hermitian-flatness of Hermitian manifolds. Let (M, J, g) be a Hermitian manifold of complex dimension m.

DEFINITION 2.1. ([5]) We call a Hermitian manifold (M, J, g) or the Hermitian metric g to be *Hermitian-flat* if the curvature tensor of the Hermitian connection with respect to g vanishes everywhere.

DEFINITION 2.2. ([5]) A Hermitian manifold (M, J, g) is *locally conformally Hermitian-flat* if every point of M has an open neighborhood U with a differentiable function $\sigma : U \to \mathbf{R}$ such that $g' = e^{-\sigma}g_{\,|\,U}$ is a Hermitian-flat metric on U

Assume that the tensor $\mathfrak{B}$ vanishes everywhere on M. Since the Ricci form ρ_R is a closed 2-form of type (1,1), on a neighborhood U of every point of M, we obtain a differentiable function σ such that $\rho_R = mdd^{\mathbf{c}}\sigma$ (cf. [1]). Then, with respect to $g' = e^{-\sigma}g_{|U}$, we have $H' = 0$ on U. Now we have

THEOREM 2.4. ([6]) *A Hermitian manifold (M, J, g) is locally conformally Hermitian-flat if and only if the tensor $\mathfrak{B}$ vanishes everywhere on M.*

On the other hand, about the vanishing of $\mathcal{B}_0$, we have

THEOREM 2.5. ([5]) *Under the assumption that $m \geq 3$, a Hermitian manifold (M, J, g) is locally conformally Hermitian-flat if and only if both the tensor $\mathcal{B}_0$ and the function $m\hat{s} - s$ vanish everywhere on M.*

From these theorems, the vanishing of $\mathfrak{B}$ means one of $\mathcal{B}_0$. Conversely the vanishing of $\mathcal{B}_0$ does not mean one of $\mathfrak{B}$. In fact, in the Kählerian case, $\mathfrak{B} = 0$ is equivalent to flatness (cf. [6]) and $\mathcal{B}_0 = 0$ is equivalent to be of constant holomorphic sectional curvature provided with $m \geq 3$ (cf. [5]). We notice that the Weyl conformal curvature tensor $W = 0$ is equivalent to flatness in the Kählerian case.

The tensor B in the curvature decomposition (2.6) is also conformally invariant. We can easily check it by the equation (2.2) and the conformal relations for the Ricci-type tensor S and the scalar curvature s (cf. [1], [3]). From comparing with this tensor B, we shall give a characterization of the tensor $\mathfrak{B}$. We see that the tensor B consists of two conformally invariant parts:

$$B = \left\{ H - \frac{1}{m}\Omega \otimes \rho_R \right\} + \left\{ -\frac{1}{m}\left(\rho_S - \frac{s}{2m}\Omega\right) \otimes \Omega \right\}. \tag{2.9}$$

Then we obtain

PROPOSITION 2.1. *The tensor $\mathfrak{B}$ is one of two conformally invariant parts of the tensor B.*

The vanishing of $\mathfrak{B}$ induces one of B because $\mathfrak{B} = 0$ implies $\rho_S = \dfrac{s}{2m}\Omega$. The converse does not hold in general. Unfortunately we know no example of Hermitian manifolds such that $B = 0$ and $\mathfrak{B} \neq 0$.

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