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# REMARKS ON M–IDEALS OF COMPACT OPERATORS

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## 1. Introduction

A closed subspace $J$ of a Banach space $X$ is called an M-ideal in $X$ if the annihilator $J^{\perp}$ of $J$ is an L-summand of $X^*$. That is, there exists a closed subspace $J'$ of $X^*$ such that $X^* = J^{\perp} \oplus J'$ and $\|p+q\| = \|p\| + \|q\|$ whenever $p \in J^{\perp}$ and $q \in J'$. Ever since Alfsen and Effros [1] introduced the notion of an M-ideal in a Banach space, many authors have studied the problem determining those Banach spaces $X$ and $Y$ for which $K(X,Y)$, the space of compact linear operators from $X$ to $Y$, is an M-ideal in $L(X,Y)$, the space of bounded linear operators from $X$ to $Y$ [3, 7, 8, 9, 11, 14, 15, 16]. It is well known that if $X$ is a Hilbert space, $\ell_p$ $(1 < p < \infty)$ or $c_0$, then $K(X)(=K(X,X))$ is an M-ideal in $L(X)(=L(X,X))$ [4, 7, 15] while $K(\ell_1)$ and $K(\ell_\infty)$ are not M-ideals in the corresponding space of operators [15]. Also several authors proved that $K(\ell_p, \ell_q)$ for $1 < p \leq q < \infty$ is an M-ideal in $L(\ell_p, \ell_q)$ [6, 11, 14] and $K(X, c_0)$ is an M-ideal in $L(X, c_0)$ for every Banach space $X$ [14, 15].

Kalton [8] introduced the notions of property (M) and property (M*), and proved that if $X$ is a separable Banach space then $K(X)$ is an M-ideal in $L(X)$ if and only if $X$ has either property (M) or property (M*), and there exists a sequence $\{K_n\}_{n=1}^{\infty}$ in $K(X)$ such that (i) $K_n \to I_X$ (the identity map on $X$) strongly, (ii) $K_n^* \to I_{X^*}$ strongly and (iii) $\|I_X - 2K_n\| \to 1$. Property (M*) is a dual notion of property (M).

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Later Kalton and Werner [9] extended the notion of property (M) to a single operator from a Banach space $X$ to a Banach space $Y$ and established a necessary and sufficient condition for $K(X,Y)$ to be an M-ideal in $L(X,Y)$. More specifically, they proved the following theorem.

THEOREM A [9]. *Suppose $X$ is a Banach space such that there exists a sequence $\{K_n\}_{n=1}^{\infty}$ in $K(X)$ satisfying*

(i) $K_n \to I_X$ *strongly,*
(ii) $K_n^* \to I_{X^*}$ *strongly,*
(iii) $\|I_X - 2K_n\| \to 1$.

*If $Y$ is a Banach space, then $K(X,Y)$ is an M-ideal in $L(X,Y)$ if and only if every $T \in L(X,Y)$ with $\|T\| \le 1$ has the property (M).*

Using this theorem Kalton and Werner [9] proved that if $2 \le p < \infty$ then $K(\ell_p, L_p[0,1])$ is an M-ideal in $L(\ell_p, L_p[0,1])$. They also mentioned about the validity of a property (M*) version of Theorem A, but did not pursue its application.

In this paper, using Theorem A we will draw a result which comprises that of Kalton and Werner. In Theorem 3.2 we will prove that if $2 \le p \le q < \infty$, $q \ne 2$, and $X$ is a closed subspace of $(\sum X_n)_p$ ($\dim X_n < \infty$) which has the compact approximation property then $K(X, L_q[0,1])$ is an M-ideal in $L(X, L_q[0,1])$. We will also formulate (M*) version of Theorem A. Unexpectedly we will see that we don't need the condition "$K_n^* \to I_{Y^*}$ strongly" in Theorem 3.3. As an application we will prove that $K(L_1[0,1], \ell_1)$ is not an M-ideal in $L(L_1[0,1], \ell_1)$ in Theorem 3.4.

## 2. Preliminaries

If $X$ is a Banach space, $B_X$ will denote the closed unit ball of $X$ and $I_X$ will denote the identity map on $X$. A Banach space $X$ is said to have the compact approximation property if the identity operator on $X$ is in the closure of $K(X)$ with respect to the topology of uniform convergence on compact sets in X.

A Banach space $X$ is said to have a finite-dimensional Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ if every $x \in X$ can be uniquely written as $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ and each $X_n$ is a finite-dimensional subspace of $X$. For each $n$ the partial sum projection $P_n$ on $X$ is defined by $P_n(\sum_{i=1}^{\infty} x_i) = \sum_{i=1}^{n} x_i$, where $x_i \in X_i$. It is easy to see that $\sup_n \|P_n\| < \infty$. A Banach space $X$ with a finite-dimensional Schauder decomposition $\{X_n\}_{n=1}^{\infty}$ is called the $\ell_p$-sum of $\{X_n\}_{n=1}^{\infty}$ and is written $X = (\sum X_n)_p$ if $\|\sum x_n\| = (\sum \|x_n\|^p)^{1/p}$ for every $x = \sum x_n \in X$ with $x_n \in X_n$.

By an operator between Banach spaces we will always mean a continuous linear operator. According to Kalton and Werner [9] an operator $T$ with $\|T\| \leq 1$ from a Banach space $X$ to a Banach space $Y$ is said to have property (M) if

$$\limsup_{n\to\infty} \|y + Tx_n\| \leq \limsup_{n\to\infty} \|x - x_n\|$$

for all $x \in X$, $y \in Y$ with $\|y\| \leq \|x\|$ and all weakly null sequences $\{x_n\}_{n=1}^{\infty}$ in $X$.

Dualizing property (M), let us say that a contractive operator $T : X \to Y$ has property (M*) if

$$\limsup_{n\to\infty} \|x^* + T^* y_n^*\| \leq \limsup_{n\to\infty} \|y^* + y_n^*\|$$

for all $x^* \in X^*$, $y^* \in Y^*$ with $\|x^*\| \leq \|y^*\|$ and all weak* null sequence $\{y_n^*\}_{n=1}^{\infty}$ in $Y^*$.

The following lemma is a property (M*) version of Lemma 2.2 of [9].

LEMMA 2.1. *If $T : X \to Y$ is a contractive operator with property (M*), and $\{x_n^*\}_{n=1}^{\infty} \subseteq X^*$ and $\{y_n^*\}_{n=1}^{\infty} \subseteq Y^*$ are relatively compact sequences with $\|x_n^*\| \leq \|y_n^*\|$ for all $n$, then*

$$\limsup_{n\to\infty} \|x_n^* + T^* z_n^*\| \leq \limsup_{n\to\infty} \|y_n^* + z_n^*\|$$

*for all weak* null sequence $\{z_n^*\}_{n=1}^{\infty}$ in $Y^*$.*

*Proof.* The proof is straight forward (or see the proof of Lemma 2.2 of [9]).

## 3. M-ideals

The key fact in the proof of Theorem 3.2 is the following.

PROPOSITION 3.1. *Suppose $X$ is a reflexive subspace of a Banach space $Z$ with the property that there exists a sequence $\{P_n\}_{n=1}^{\infty}$ in $K(Z)$ such that $\limsup_{n\to\infty} \|I_Z - 2P_n\| \leq 1$ and $P_n \to I_Z$ strongly. If $X$ has the compact approximation property, then there exists a sequence $\{K_n\}_{n=1}^{\infty}$ in $B_{K(X)}$ satisfying*

(i) $K_n \to I_X$ *strongly,*
(ii) $K_n^* \to I_{X^*}$ *strongly,*
(iii) $\|I_X - 2K_n\| \to 1$.

*Proof.* From the Cho–Johnson proof of Proposition 3 of [3] we can find a sequence $\{K_n\}_{n=1}^{\infty}$ in $B_{K(X)}$ satisfying (i), (ii) and

$$\limsup_{n\to\infty} \|I_X - 2K_n\| \leq 1.$$

Now fix $x \in X$ with $\|x\| = 1$. Since $K_n x \to x$ and $\|I_X - 2K_n\| \geq \|x - 2K_n x\|$, we have

$$\liminf_{n\to\infty} \|I_X - 2K_n\| \geq \|x\| = 1.$$

Therefore, $\|I_X - 2K_n\| \to 1$.

THEOREM 3.2. *Suppose $2 \leq p \leq q < \infty$ and $q \neq 2$. If $X$ is a closed subspace of $(\sum X_n)_p$ $(\dim X_n < \infty)$ which has the compact approximation property, then $K(X, L_q[0,1])$ is an M-ideal in $L(X, L_q[0,1])$.*

*Proof.* By Proposition 3.1, there exists a sequence $\{K_n\}_{n=1}^{\infty}$ in $K(X)$ satisfying condition (i), (ii) and (iii) in Theorem A. Therefore, it suffices to prove that every contraction in $L(X, L_q[0,1])$ has property (M).

Let $T : X \to L_q[0,1]$ be an operator with $\|T\| \leq 1$. Let $x \in X$, $y \in L_q[0,1]$ with $\|y\| \leq \|x\|$ and let $\{x_n\}_{n=1}^{\infty}$ be a weakly null sequence in $X$. We need to prove

$$\limsup_{n\to\infty} \|y + Tx_n\| \leq \limsup_{n\to\infty} \|x + x_n\|.$$

If $\|Tx_n\| \to 0$, then there is nothing to prove. So we assume that $\limsup_{n\to\infty} \|Tx_n\| > 0$. By passing to subsequences we may assume that $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis $\{e_n\}_{n=1}^{\infty}$ of $\ell_p$ and $\{\|y + Tx_n\|\}_{n=1}^{\infty}$ converges. Hence there exists $\alpha > 0$ such that

$$\left\| \sum_{n=1}^{N} \pm e_n \right\| \geq \alpha \left\| \sum_{n=1}^{N} \pm x_n \right\|$$

for all choices of signs and every positive integer $N$. By the Kalton-Werner argument [9, Proposition 2.5], we can prove that $\{Tx_n\}_{n=1}^{\infty}$ is uniformly integrable. Hence we have

$$\begin{aligned} \limsup_{n\to\infty} \|y + Tx_n\| &= \limsup_{n\to\infty} \left( \|y\|^q + \|Tx_n\|^q \right)^{1/q} \\ &\leq \limsup_{n\to\infty} \left( \|x\|^q + \|x_n\|^q \right)^{1/q} \\ &\leq \limsup_{n\to\infty} \left( \|x\|^p + \|x_n\|^p \right)^{1/p}. \end{aligned}$$

Since $x_n \to 0$ weakly, $\limsup_{n\to\infty} \|x + x_n\|^p$ $= \limsup_{n\to\infty} \left( \|x\|^p + \|x_n\|^p \right)^{1/p}$ and our proof is complete.

REMARKS. 1. Observe that we can replace $L_q[0,1]$ by any subspace $Y$ of $L_q[0,1]$ in Theorem 3.2.

2. An M-ideal is a proximinal subspace and hence Theorem 3.2 gives an answer for $q > 2$ to the question raised by Bang and Odel [2] asking whether $K(\ell_2, L_p[0,1])$ is an M-ideal in $L(\ell_2, L_p[0,1])$ if $1 < p < \infty$.

Alfsen and Effros [1], and Lima [10] characterized an M-ideal in terms of intersection properties of balls. Lima [10, Theorem 6.17] proved that a closed subspace $J$ of a Banach space $X$ is an M-ideal in $X$ if and only if for any $x_1, x_2, x_3 \in B_J$, $x \in B_X$ and any $\varepsilon > 0$, there exists $y \in J$ such that $\|x_i + x - y\| < 1 + \varepsilon$ for $i = 1, 2, 3$.

Now we will formulate a property (M*) version of Theorem A. The following proof of Theorem 3.3 is a minor modification of the Kalton-Werner proof of Theorem A [9, Theorem 2.3]. Since we don't need the condition "$K_n^* \to I_{Y^*}$ strongly" in Theorem 3.3, it is applicable in the case that $Y^*$ is nonseparable.

THEOREM 3.3. *Suppose $Y$ is a Banach space such that there exists a sequence $\{K_n\}_{n=1}^{\infty}$ in $K(Y)$ satisfying*

(i) *$K_n \to I_Y$ strongly,*

(ii) *$\|I_Y - 2K_n\| \to 1$.*

*If $X$ is a Banach space, then $K(X, Y)$ is an M-ideal in $L(X, Y)$ if and only if every contractive operator $T : X \to Y$ has property ($M^*$).*

*Proof.* Suppose that every contractive operator in $L(X, Y)$ has property ($\mathrm{M}^*$). Let $S_i \in B_{K(X,Y)}$ $(i = 1, 2, 3)$, $T \in B_{L(X,Y)}$ and $\varepsilon > 0$ be given. We will show that there exists $S \in K(X, Y)$ such that

$$\|S_i + T - S\| \leq 1 + \varepsilon \quad \text{for} \quad i = 1, 2, 3.$$

Choose $m$ so that $\|S_i - K_m S_i\| < \frac{\varepsilon}{2}$ $(i = 1, 2, 3)$ and $\|I_Y - 2K_m\| < 1 + \frac{\varepsilon}{2}$. Choose a sequence $\{y_n^*\}_{n=1}^{\infty}$ in $Y^*$ such that $\|y_n^*\| = 1$ for all $n$ and

$$\begin{aligned}\limsup_{n\to\infty} \|K_m S_1 + T - K_n T\| &= \limsup_{n\to\infty} \|S_1^* K_m^* + T^* - T^* K_n^*\| \\ &= \limsup_{n\to\infty} \|S_1^* K_m^* y_n^* + T^*(I_{Y^*} - K_n^*) y_n^*\| .\end{aligned}$$

Since $\{(I_{Y^*} - K_n^*)y_n^*\}_{n=1}^{\infty}$ is a weak* null sequence in $Y^*$, and $\{K_m^* y_n^*\}_{n=1}^{\infty} \subseteq Y^*$ and $\{S_1^* K_m^* y_n^*\}_{n=1}^{\infty} \subseteq X^*$ are relatively compact sequecnes with $\|S_1^* K_m^* y_n^*\| \leq \|K_m^* y_n^*\|$ for all $n$, by Lemma 2.1 we have

$$\begin{aligned}&\limsup_{n\to\infty} \|S_1^* K_m^* y_n^* + T^*(I_{Y^*} - K_n^*) y_n^*\| \\ &\qquad \leq \limsup_{n\to\infty} \|K_m^* y_n^* + (I_{Y^*} - K_n^*) y_n^*\| \\ &\qquad \leq \limsup_{n\to\infty} \|K_n + I_Y - K_n\| \\ &\qquad \leq \|I_Y - 2K_m\| \quad [6, \text{p.}300] \\ &\qquad < 1 + \frac{\varepsilon}{2}.\end{aligned}$$

Combining above inequalities, we have

$$\limsup_{n\to\infty} \|S_1 + T - K_n T\| < 1 + \varepsilon.$$

Thus $\lim_{j\to\infty} \left\|S_1 + T - K_{n_j} T\right\| < 1+\varepsilon$ for some subsequence $\left\{K_{n_j}\right\}_{j=1}^{\infty}$ of $\{K_n\}_{n=1}^{\infty}$.

Repeating the same argument using $\left\{K_{n_j}\right\}_{j=1}^{\infty}$ in place of $\{K_n\}_{n=1}^{\infty}$, we have the same type of inequalities for $S_2$, $S_3$ and for some subsequence of $\left\{K_{n_j}\right\}_{j=1}^{\infty}$. Therefore, $K(X,Y)$ is an M-ideal in $L(X,Y)$.

Conversely, suppose $K(X,Y)$ is an M-ideal in $L(X,Y)$ and $T \in L(X,Y)$ is a contraction. We will show that $T$ has property (M*).

Let $x^* \in X^*$, $y^* \in Y^*$ with $\|x^*\| \le \|y^*\| \le 1$ and $\{y_n^*\}_{n=1}^{\infty}$ a weak* null sequence in $Y^*$. Let $\varepsilon > 0$ be given. Choose $y \in Y$ with $\|y\| = 1$ such that $1 - \varepsilon \le y^*(y) \le 1$. Let $S = x^* \otimes y$. Then $\|S\| = \|x^*\| \le 1$ and $S^* = \hat{y} \otimes x^*$, where $\hat{y}$ is the natural embedding of $y \in Y$ into $Y^{**}$.

Since $S^*(y^*) = y^*(y)x^*$, $\|S^*y^* - x^*\| \le \varepsilon$ and

$$\limsup_{n\to\infty} \|x^* + T^*y_n^*\| \le \varepsilon + \limsup_{n\to\infty} \|S^*y^* + T^*y_n^*\| .$$

Choose $U \in K(X,Y)$ such that $\|(T^* - U^*)y^*\| < \varepsilon$ and $\|S + T - U\| < 1 + \varepsilon$ [17, Theorem 3.1, Remark].

Since $S$ is compact and $\{y_n^*\}_{n=1}^{\infty}$ is a uniformly bounded weak* null sequence, $y_n^* \to 0$ uniformly on $S(B_X)$ and hence $\|S^*y_n^*\| \to 0$. Similarly, $\|U^*y_n^*\| \to 0$. Thus

$$\begin{aligned}\limsup_{n\to\infty} \|S^*y^* + T^*y_n^*\| &= \limsup_{n\to\infty} \|S^*(y^* + y_n^*) + (T^* - U^*)y_n^*\| \\ &\le \varepsilon + \limsup_{n\to\infty} \|(S^* + T^* - U^*)(y^* + y_n^*)\| \\ &\le \varepsilon + (1+\varepsilon) \limsup_{n\to\infty} \|y^* + y_n^*\| .\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have
$\limsup_{n\to\infty} \|x^* + T^*y_n^*\| \le \limsup_{n\to\infty} \|y^* + y_n^*\|$.

THEOREM 3.4. *$K(L_1[0,1], \ell_1)$ is not an M-ideal in $L(L_1[0,1], \ell_1)$.*

*Proof.* In view of Theorem 3.3, it suffices to show that there is a contraction $T : L_1[0,1] \to \ell_1$ which does not have property (M*).

Partition $[0,1)$ into $\{I_n\}_{n=1}^{\infty}$, where

$$I_n = \left[\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n - 1}{2^n}\right) \quad n \ge 1.$$

For $f \in L_1[0,1]$ we write $f = \sum_{n=1}^{\infty} f\chi_{I_n}$ and define $Tf = \sum_{n=1}^{\infty} \left(\frac{1}{m(I_n)} \int_{I_n} f\right) \chi_{I_n}$, where $m$ is the Lebesgue measure on $[0,1]$ and $\chi_{I_n}$ is the characteristic function of $I_n$. Then $T$ is a norm one projection on $L_1[0,1]$ and the map $\frac{1}{m(I_n)}\chi_{I_n} \to e_n$ gives an isometry between the range of $T$ and $\ell_1$, where $\{e_n\}_{n=1}^{\infty}$ is the unit vector basis of $\ell_1$.

We claim that $T$ dose not have property (M*). Let $\{\hat{e}_n\}_{n=1}^{\infty}$ be the unit vector basis of $c_0$ canonically embedded in $\ell_\infty$. Put $x^* = \chi_{[0,1]} \in L_1[0,1]$ and $y^* = \hat{e}_1 \in \ell_1$. Since $\{\hat{e}_n\}_{n=1}^{\infty}$ is a weak* null sequence and $\|T^*\hat{e}_n\| \not\to 0$, we have

$$\limsup_{n\to\infty} \|x^* + T^*(\theta_n \hat{e}_n)\| > \limsup_{n\to\infty} \|y^* + \theta_n \hat{e}_n\|$$

for appropriate choices of $\theta = \pm 1$. Hence $T$ does not have property (M*).

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