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CHARACTERIZATION OF BEST SIMULTANEOUS APPROXIMATION FOR A COMPACT SET

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Let X be a normed linear space and K be a nonempty subset of X. For any subset F of X, we define

$$d(F,K) := \inf_{k\in K} \sup_{f\in F} ||f-k||$$

and the elements in K which attain the above infimum are called the best simultaneous approximations for F from K.

Throughout this article, we assume that X is a real normed linear space and K is a nonempty subset of X.

For each positive integer n, define the set

$$\bar{F}_n := \{(\bar{\lambda}_n, \bar{f}_n) \mid \bar{\lambda}_n = (\lambda_1, \dots, \lambda_n), \bar{f}_n = (f_1, \dots, f_n), \\ f_i \in F, \lambda_i \geq 0\,(i = 1, \dots, n), \sum_{i=1}^{n} \lambda_i = 1\}.$$

Let U and V be nonempty compact convex subsets of two Hausdorff topological vector spaces. Suppose that a function $J : U \times V \to \mathbb{R}$ is such that for each $v \in V, J(\cdot, v)$ is lower semi-continuous and convex on U and for each $u \in U, J(u, \cdot)$ is upper semi-continuous and concave on V. Then, as is well known [1], there exists a saddle point $(u^*, v^*) \in U \times V$ such that

$$J(u^*, v) \leq J(u^*, v^*) \leq J(u, v^*),\ u \in U, v \in V,$$

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that is,

$$\min_{u\in U}\max_{v\in V} J(u,v) = \max_{v\in V}\min_{u\in U} J(u,v).$$

However, if the set V is not convex or if for some $u \in U$, $J(u,\cdot)$ is not a concave function on V, then the above relation does not hold in general.

THEOREM 1 [5]. *Let U be an n-dimensional, compact convex subset of a Hausdorff topological vector space ($n \geq 1$), and let V be a compact Hausdorff space. Let $J : U \times V \to \mathbb{R}$ be a jointly continuous function. Then $u^* \in U$ minimizes $\max_{v\in V} J(u,v)$ over U if and only if there exists $(\lambda^*_{n+1}, \bar{v}^*_{n+1}) \in \bar{V}_{n+1}$ such that*

$$\sum_{i=1}^{n+1} \lambda_i J(u^*, v_i) \leq \sum_{i=1}^{n+1} \lambda_i^* J(u^*, v_i^*) \leq \sum_{i=1}^{n+1} \lambda_i^* J(u, v_i^*)$$

holds for all $(\lambda_{n+1}, \bar{v}_{n+1}) \in \bar{V}_{n+1}$ and for all $u \in U$.

We firstly study the existence of a best simultaneous approximation. We have the following lemma.

LEMMA 2 [4]. *Suppose that K is a closed convex subset of a finite-dimensional subspace of a normed linear space X. For any compact subset $F \subset X$, there exists a best simultaneous approximation for F from K.*

The main theorem of this article is the following.

THEOREM 3. *Suppose that K is a closed convex subset of an n-dimensional subspace of X and let F be a compact subset of X. Then $k_o \in K$ is a best simultaneous approximation for F from K if and only if there exist $f_1^*, \dots, f_p^* \in F$ and positive real numbers $\lambda_1^*, \dots, \lambda_p^*$ with $\sum_{i=1}^{p} \lambda_i^* = 1$ satisfying*

(1) $||f_i^* - k_o|| = \max_F ||f - k_o||, \quad i = 1, \dots, p$,

(2) $\sum_{i=1}^{p} \lambda_i^* ||f_i^* - k_o|| \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^* - k||$ *for any* $k \in K$,

for some $1 \leq p \leq n+1$.

Proof. Let $k_o \in K$ be a best simultaneous approximation for F from K, and let $U = \{k \in K : ||k_o - k|| \leq 1\}$. Note that U is a compact convex subset of K. Define a map J from $U \times F$ to $\mathbb{R}$ by $(u, f) \mapsto ||f - u||$. Then J is jointly continuous. By Theorem 1, $k_o \in U$ minimizes $\max_F ||f - u||$ over U if and only if there exists $(\bar{\lambda}^*_{n+1}, \bar{f}^*_{n+1}) \in \bar{F}_{n+1}$ such that

$$(1.1) \qquad \sum_{i=1}^{n+1} \lambda_i ||f_i - k_o|| \leq \sum_{i=1}^{n+1} \lambda_i^* ||f_i^* - k_o|| \leq \sum_{i=1}^{n+1} \lambda_i^* ||f_i^* - u||$$

holds for all $(\bar{\lambda}_{n+1}, \bar{f}_{n+1}) \in \bar{F}_{n+1}$ and for all $u \in U$. By reindexing, we assume that $\lambda_1^*, \dots, \lambda_p^*$ are the nonzero elements within $\{\lambda_i^*\}_{i=1}^{n+1}$ and by $f_1^*, \dots, f_p^*$ the corresponding elements within $\{f_i^*\}_{i=1}^{n+1}$. Since the above inequality is true for all $(\bar{\lambda}_{n+1}, \bar{f}_{n+1})$,

$$||f_i^* - k_o|| = \max_F ||f - k_o||, \quad i = 1, \dots, p.$$

And, by the second inequality in (1.1), the convex function $u \mapsto \sum_{i=1}^{p} \lambda_i^* ||f_i^* - u||$ has a local minimum at k_o over U. Thus k_o realizes a global minimum, by a property of a convex function.

Conversely, since for any $k \in K$, $\sum_{i=1}^{p} \lambda_i^* ||f_i^* - k_o|| \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^* - k||$,

$$\begin{aligned} \max_F ||f - k_o|| &\leq \max_{\bar{F}_p} \inf_K \sum_{i=1}^{p} \lambda_i ||f_i - k|| \\ &\leq \inf_K \max_{\bar{F}_p} \sum_{i=1}^{p} \lambda_i ||f_i - k|| \\ &= \inf_K \max_F ||f - k||. \end{aligned}$$

Thus k_o is a best simultaneous approximation for F from K. $\square$

We can rewrite Theorem 3 in the following precise form. If $k_o \in K$ is a best simultaneous approximation for F if and only if there exist

$f_1^*, \dots, f_p^* \in F$ and positive real numbers $\lambda_1^*, \dots, \lambda_p^*$ with $\sum_{i=1}^{p} \lambda_i^* = 1$ such that

(1) $||f_i^* - k_o|| = d(F, K), \quad i = 1, \dots, p,$

(2) $d(F, K) \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^* - k||$ for any $k \in K$,

for some $1 \leq p \leq n+1$.

Since each finite set is compact, we obtain the following corollary.

COROLLARY 4 [3]. *Let K be a closed convex subset of an n-dimensional subspace of a normed linear space X and $x_1, \dots, x_\ell \in X$. Then $k_o \in K$ is a best simultaneous approximation for $\{x_1, \dots, x_\ell\}$ from K if and only if there exist positive real numbers $\lambda_1^*, \dots, \lambda_p^*$ with $\sum_{i=1}^{p} \lambda_i^* = 1$ and p vectors $\mathrm{a}_1^*, \dots, \mathrm{a}_p^* \in A$ for some $1 \leq p \leq n+1$ such that*

(1) $|| \sum_{j=1}^{\ell} a_{ij}^* x_j - k_o|| = \max_{1 \leq j \leq \ell} ||x_j - k_o||, \quad i = 1, \dots, p,$

(2) $\sum_{i=1}^{p} \lambda_i^* || \sum_{j=1}^{\ell} a_{ij}^* x_j - k_o|| \leq \sum_{i=1}^{p} \lambda_i^* || \sum_{j=1}^{\ell} a_{ij}^* x_j - k||$ *for any $k \in K$,*

where the set A is defined by

$$A := \{\mathrm{a} = (a_1, \dots, a_\ell) | \sum_{j=1}^{\ell} a_j = 1, a_j \geq 0 \ \text{ for } j = 1, \dots, \ell\}.$$

Let S be a compact Hausdorff space, and let T be a real normed linear space with the norm $|| \cdot ||$. Suppose that $C(S, T)$ is the set of all continuous functions from S to T and let K be a closed convex subset of an n-dimensional subspace in $C(S, T)$. For $f \in C(S, T)$, we define the uniform norm of f by

$$|||f||| = \max_{s \in S} ||f(s)||$$

and endow the linear space $C(S, T)$ with the uniform topology. Suppose that F is a compact subset of $C(S, T)$. We want to approximate the compact subset F simultaneously by functions in K. That is, we want to find a function $k^* \in K$ which minimizes

$$\max_F |||f-k||| = \max_F \max_S ||f(s)-k(s)||$$

over K. If such a function k^* exists in K, we call it a best simultaneous (uniform) approximation for F. Thus

$$\begin{aligned}\max_F |||f-k^*||| &= \max_{\bar{F}_n} \sum_{i=1}^n \lambda_i |||f_i - k^*||| \\ &= \max_{\bar{F}_n} \max_S \sum_{i=1}^n \lambda_i ||f_i(s) - k^*(s)||.\end{aligned}$$

So

$$\min_K \max_{\bar{F}_n} ||| \sum_{i=1}^n \lambda_i f_i - k||| = \min_K \max_{\bar{F}_n \times S} || \sum_{i=1}^n \lambda_i f_i(s) - k(s)||.$$

Note that the set $\bar{F}_n \times S$ is compact. Then

$$|| \sum_{i=1}^n \lambda_i f_i(s) - k(s)||$$

is jointly continuous with respect to $\bar{\lambda}_n, s, k$ and convex in k.

THEOREM 5. *Suppose that K is a closed convex subset of an n-dimensional subspace in $C(S,T)$ and let F be a compact subset of $C(S,T)$. Then $k^* \in K$ is a best simultaneous approximation for F from K if and only if there exist $f_1^*, \ldots, f_p^* \in F$, $s_1^*, \ldots, s_p^* \in S$ and positive real numbers $\lambda_1^*, \ldots, \lambda_p^*$ with $\sum_{i=1}^p \lambda_i^* = 1$ such that*

(1) $||f_i^*(s_i^*) - k^*(s_i^*)|| = \max_F |||f-k^*|||, \quad i = 1, \ldots, p,$

(2) $\sum_{i=1}^p \lambda_i^* ||f_i^*(s_i^*) - k^*(s_i^*)|| \le \sum_{i=1}^p \lambda_i^* ||f_i^*(s_i^*) - k(s_i^*)||$ *for any* $k \in K$,

for some $1 \leq p \leq n+1$.

Proof. Let k^* be a best simultaneous approximation for F, and let $U = \{k \in K : |||k^* - k||| \leq 1\}$. Define $J : U \times (F \times S) \to \mathbb{R}$ by $(u, f, s) \mapsto ||f(s) - u(s)||$. Then J is a jointly continuous function. By Theorem 1, $k^* \in U$ minimizes $\max_{F \times S} ||f(s) - k(s)||$ over U if and only if there exists $(\bar{\lambda}^*_{n+1}, \bar{f}^*_{n+1}, \bar{s}^*_{n+1}) \in \overline{(F \times S)}_{n+1}$ such that

$$\begin{aligned}\sum_{i=1}^{n+1} \lambda_i ||f_i(s_i) - k^*(s_i)|| &\leq \sum_{i=1}^{n+1} \lambda^*_i ||f^*_i(s^*_i) - k^*(s^*_i)|| \\ &\leq \sum_{i=1}^{n+1} \lambda^*_i ||f^*_i(s^*_i) - k(s^*_i)||\end{aligned}$$

holds for all $(\bar{\lambda}_{n+1}, \bar{f}_{n+1}, \bar{s}_{n+1}) \in \overline{(F \times S)}_{n+1}$ and for all $k \in U$ where

$$\begin{aligned}\overline{(F \times S)}_{n+1} = \{(\bar{\lambda}_{n+1}, \bar{f}_{n+1}, \bar{s}_{n+1}) \mid & \bar{f}_{n+1} = (f_1, \ldots, f_{n+1}), f_i \in F, \\ & \bar{s}_{n+1} = (s_1, \ldots, s_{n+1}), s_i \in S, \bar{\lambda}_{n+1} = (\lambda_1, \ldots, \lambda_{n+1}), \\ & \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0 \, (i = 1, \ldots, n+1)\}.\end{aligned}$$

Let us denote by $\lambda^*_1, \ldots, \lambda^*_p$ the nonzero elements within $\{\lambda^*_1, \ldots, \lambda^*_{n+1}\}$ and by $f^*_1, \ldots, f^*_p$ and $s^*_1, \ldots, s^*_p$ the corresponding elements within $\{f^*_1, \ldots, f^*_{n+1}\}$ and $\{s^*_1 \ldots, s^*_{n+1}\}$, respectively. So

$$\sum_{i=1}^{p} \lambda^*_i ||f^*_i(s^*_i) - k^*(s^*_i)|| = d(F, K).$$

Thus, for all $i = 1, \ldots, p$,

$$\begin{aligned}||f^*_i(s^*_i) - k^*(s^*_i)|| &= \max_{F \times S} ||f(s) - k^*(s)|| \\ &= \max_{F} |||f - k^*||| = d(F, K).\end{aligned}$$

Since $\sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - (\cdot)(s_i^*)||$ is a convex function and has a local minimum at k^* over U, k^* realizes a global minimum over K.

Conversely, suppose that (1) and (2) of Theorem 5 hold.

$$\begin{aligned}
\max_F |||f - k^*||| &= \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k^*(s_i^*)|| \\
&\leq \inf_K \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k(s_i^*)|| \\
&\leq \max_{(F\times S)_p} \inf_K \sum_{i=1}^{p} \lambda_i ||f_i(s_i) - k(s_i)|| \\
&\leq \inf_K \max_{(F\times S)_p} \sum_{i=1}^{p} \lambda_i ||f_i(s_i) - k(s_i)|| \\
&\leq \inf_K \max_F |||f - k|||.
\end{aligned}$$

So k^* is a best simultaneous approximation for F from K. □

By Theorem 3 and Theorem 5, we have the following result.

COROLLARY 6. *Suppose that K is a closed convex subset of an n-dimensional subspace in $C(S,T)$ and let F be a compact subset of $C(S,T)$. Then the following statements are equivalent:*

(1) *$k^* \in K$ is a best simultaneous approximation for F from K.*

(2) *There exist $f_1^*, \dots, f_p^* \in F$, $s_1^*, \dots, s_p^* \in S$ and positive real numbers $\lambda_1^*, \dots, \lambda_p^*$ with $\sum_{i=1}^{p} \lambda_i^* = 1$ such that*

(a) $||f_i^*(s_i^*) - k^*(s_i^*)|| = \max_F |||f - k^*|||$, $i = 1, \dots, p$,

(b) $\sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k^*(s_i^*)|| \leq \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k(s_i^*)||$ *for any* $k \in K$,

for some $1 \leq p \leq n+1$.

(3) *There exist $f_1^*, \dots, f_q^* \in F$ and positive real numbers $\lambda_1^*, \dots, \lambda_q^*$ with $\sum_{i=1}^{q} \lambda_i^* = 1$ such that*

(a) $|||f_i^* - k^*||| = \max_F |||f - k^*|||, \quad i = 1, \ldots, q,$

(b) $\sum_{i=1}^{q} \lambda_i^* |||f_i^* - k^*||| \le \sum_{i=1}^{q} \lambda_i^* |||f_i^* - k|||$ *for any* $k \in K$,

for some $1 \le q \le n+1$.

If T is a Hilbert space with an inner product $< \cdot >$, then the condition (2) of Theorem 5 can be replaced by another form as follows.

COROLLARY 7. *Suppose that T is a Hilbert space with an inner product $< \cdot >$ and K is a closed convex subset of an n-dimensional subspace in $C(S, T)$. Let F be a compact subset of $C(S, T)$. Then $k_o \in K$ is a best simultaneous approximation for F from K if and only if there exist $f_1^*, \ldots, f_p^* \in F$, $s_1^*, \ldots, s_p^* \in S$ and positive real numbers $\lambda_1^*, \ldots, \lambda_p^*$ with $\sum_{i=1}^{p} \lambda_i^* = 1$ such that*

(1) $||f_i^*(s_i^*) - k_o(s_i^*)|| = \max_F |||f - k_o|||, \quad i = 1, \ldots, p,$

(2) $\sum_{i=1}^{p} \lambda_i^* \tau_+(f_i^*(s_i^*) - k_o(s_i^*), k_o(s_i^*) - k(s_i^*)) \ge 0$ *for any* $k \in K$,

where $1 \le p \le n+1$ *and* $\tau_+(\cdot, \cdot)$ *is the Gateaux derivative.*

Proof. For any $k \in K$,

$$\sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k_o(s_i^*)|| \le \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k(s_i^*)||$$

if and only if

$$(1.2) \quad \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k_o(s_i^*)|| \le \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - tk(s_i^*) - (1-t)k_o(s_i^*)||$$

for all $t \in [0, 1]$. This means that the right hand side of (1.2) is a convex function, with respect to t and has a minimum at $t = 0$. Thus, for any

$k \in K$,

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i^* \, \tau_+(f_i^*(s_i^*) - k_o(s_i^*), k_o(s_i^*) - k(s_i^*)) \\
&= \sum_{i=1}^{p} \lambda_i^* \lim_{t \to 0+} \frac{||f_i^*(s_i^*) - tk(s_i^*) - (1-t)k_o(s_i^*)|| - ||f_i^*(s_i^*) - k_o(s_i^*)||}{t} \\
&= \sum_{i=1}^{p} \lambda_i^* \lim_{t \to 0+} \frac{||f_i^*(s_i^*) - tk(s_i^*) - (1-t)k_o(s_i^*)||^2 - ||f_i^*(s_i^*) - k_o(s_i^*)||^2}{t\,(||f_i^*(s_i^*) - tk(s_i^*) - (1-t)k_o(s_i^*)|| + ||f_i^*(s_i^*) - k_o(s_i^*)||)} \\
&= \sum_{i=1}^{p} \lambda_i^* \frac{< f_i^*(s_i^*) - k_o(s_i^*), k_o(s_i^*) - k(s_i^*) >}{||f_i^*(s_i^*) - k_o(s_i^*)||} \\
&\geq 0
\end{aligned}$$

is a necessary and sufficient condition for (1.2). □

REMARK. If K is an n-dimensional subspace of $C(S,T)$, then the condition (2) of Theorem 5 can be rewritten $\sum_{i=1}^{p} \lambda_i^* < f_i^*(s_i^*) - k_o(s_i^*), k(s_i^*) >= 0$ for any $k \in K$.

An n-dimensional subspace $K \subset C(S,T)$ is said to be a Haar subspace if for any n distinct elements $\{s_1, \ldots, s_n\} \subset S$ and $\{t_1, \ldots, t_n\} \subset T$, there exists a unique $k \in K$ such that $k(s_i) = t_i$, $i = 1, \ldots, n$.

COROLLARY 8. *Suppose that S is a compact Hausdorff space that contains more than n points. Let K be an n-dimensional Haar subspace of $C(S,T)$ and let F be a compact subset of $C(S,T)$ such that F is not a singleton subset of K. Then $k^* \in K$ is a best simultaneous approximation for F from K if and only if there exist $f_1^*, \ldots, f_{n+1}^* \in F$, $s_1^*, \ldots, s_{n+1}^* \in S$ and positive real numbers $\lambda_1^*, \ldots, \lambda_{n+1}^*$ with $\sum_{i=1}^{n+1} \lambda_i^* = 1$ such that*

(1) $||f_i^*(s_i^*) - k^*(s_i^*)|| = \max_F |||f - k^*|||$, $i = 1, \ldots, n+1$.

(2) $\sum_{i=1}^{n+1} \lambda_i^* ||f_i^*(s_i^*) - k^*(s_i^*)|| \leq \sum_{i=1}^{n+1} \lambda_i^* ||f_i^*(s_i^*) - k(s_i^*)||$ *for any* $k \in K$.

Proof. If the number p in Theorem 5 is less than n+1, then there exists a unique $\tilde{k} \in K$ such that

$$\tilde{k}(s_i^*) = f_i^*(s_i^*), \quad i = 1, \dots, p.$$

Then, by (2) of Theorem 5,

$$\begin{aligned} \max_F |||f - k^*||| &= \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - k^*(s_i^*)|| \\ &\le \sum_{i=1}^{p} \lambda_i^* ||f_i^*(s_i^*) - \tilde{k}(s_i^*)|| = 0. \end{aligned}$$

Since $\max_F |||f - k^*||| > 0$, it is a contradiction. Hence $p = n + 1$. □

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