

SPECTRA AND NUMERICAL RANGES

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1. Introduction

The theory of numerical ranges in unital normed algebras has extensively studied by many authors, for example, see [1], [2] for details.

Let x be a fixed element of a non unital normed algebra A over a field $K = \mathbb{R}$ or \mathbb{C} . In [8], Yang introduced the notion of right(left) relative numerical range $V_x^R(A, a)(V_x^L(A, a))$ of an element a of a non unital normed algebra A relative to $x \in A$ (see Definition 2.1). If $x = e$, the identity of A and $\|e\| = 1$, then $V_x^R(A, a)$ coincides with $V(a)$ where $V(a)$ denotes the (Bonsall and Duncan) numerical range of a ([2]). Since x can be arbitrarily chosen, this concept extends the familiar concept of numerical range. Among the results, it is shown ([8]) that our numerical range is a compact convex subset of K .

In this paper, we introduce the notion of w-proximinal subspace of a normed linear space and show that every proper closed left or right ideal of a unital Banach algebra is w-proximinal. Also we give an example which is not a w-proximinal subspace.

It is shown ([1]) that if A is a unital Banach algebra, then $Sp_A(a) \subset V(a)$. Further, we introduce the notions of proximinal properties of a normed algebra, and for a normed algebra A having proximinal property, we give the similar inclusion relation between the spectrum $Sp_A(a)$ and the relative numerical range of $a \in A$. In particular we show that if A is a normed algebra having the first proximinal property or second proximinal property and $a \in A$, then $\{0\}^c \cap Sp_A(a) \subset V_x(A, a)$ for some $x \in A$ with $\|x\| = 1$.

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2. Spectra and relative numerical ranges

In this section, we introduce the concept of proximinal property of a normed algebra and study the inclusion relation between the spectrum and the relative numerical range.

DEFINITION 2.1. ([8]) Let A be a normed algebra over the field $K = \mathbb{R}$ or \mathbb{C} , and A' its dual. For $x \in A$, we write

$$D(A, x) = \{f \in A' : \|f\| = 1, f(x) = \|x\|\}.$$

The right relative numerical range of $a \in A$ relative to x is defined to be $V_x^R(A, a) = \{f(ax) : f \in D(A, x)\}$. The left relative numerical range of $a \in A$ relative to x is defined to be $V_x^L(A, a) = \{f(xa) : f \in D(A, x)\}$. The relative numerical range of a relative to x is defined to be $V_x(A, a) = V_x^R(A, a) \cup V_x^L(A, a)$. The right relative numerical radius of a relative to x is defined by $v_x^R(a) = \sup\{|\lambda| : \lambda \in V_x^R(A, a)\}$. The left relative numerical radius of a relative to x is defined by $v_x^L(a) = \sup\{|\lambda| : \lambda \in V_x^L(A, a)\}$. The relative numerical radius of a relative to x is defined by $v_x(a) = \max\{v_x^R(a), v_x^L(a)\}$.

Note that the set $D(A, x)$ is a nonempty subset of A' by the Hahn-Banach Theorem, and so $V_x^R(A, a)$ and $V_x^L(A, a)$ are nonempty. If A is commutative, then $V_x^R(A, a) = V_x^L(A, a) = V_x(A, a)$ as $f(ax) = f(xa)$. If $x = e$ (identity of A) with $\|e\| = 1$, then $V_e(A, a) = V(a)$, where $V(a)$ denotes the (Bonsall and Duncan) numerical range of a ([2]). Thus the concept of relative numerical range extends that of the (Bonsall and Duncan) numerical range.

LEMMA 2.2. ([8]) Let a, x be elements of a normed algebra A . Then

- (1) $D(A, x)$ is a weak* compact convex subset of A' .
- (2) $V_x^R(A, a)$ and $V_x^L(A, a)$ are compact convex subsets of K , hence $V_x(A, a)$ is a compact subset of K .
- (3) If B is a subalgebra of a normed algebra A and $b, x \in B$, then $V_x(B, b) = V_x(A, b)$.

We introduce the concept of a w-proximinal subspace in the following:

DEFINITION 2.3. Let X be a normed linear space and M a proper subspace of X . M is called w-proximinal if there exists an element $z \in X$ such that $1 = \|z\| = d(z, M)$.

It is easy to show that every proximal subspace of a normed space is w-proximinal. But it is unknown whether the converse holds or not.

EXAMPLE 2.4. (1) Let $X = \mathbb{R}^2$ and let M be the x -axis. Then M is clearly a proper subspace of a normed linear space X , and $1 = \|x\| = d(x, M)$ for some $x = (0, 1) \in X$.

(2) Let $X = l^\infty$ be the set of all bounded sequences and let $M = c_0$ be the set of all sequences that converge to 0. Then M is a closed subspace of X and hence is a Banach space(not reflexive). Also $1 = \|x\| = d(x, c_0)$ for some $x = \{1 - \frac{1}{n}\}$ in X . Hence c_0 is a w-proximinal subspace of l^∞ .

LEMMA 2.5. ([4]) Suppose X is a normed linear space over \mathbb{R} or \mathbb{C} . Let M be a finite dimensional proper subspace of X . Then there exists an $x \in X$ such that $1 = \|x\| = d(x, M)$ where $d(x, M)$ is the distance from x to M .

From the above lemma, every finite dimensional proper subspace of a normed linear space is w-proximinal.

EXAMPLE 2.6. Let $L : C[0, 1] \longrightarrow K$ be defined by

$$L(f) = \int_0^{1/2} f(x)dx - \int_{1/2}^1 f(x)dx.$$

Then $\|L\| = 1$ and $d(f, \ker L) = |L(f)|$ by ([7], problem 3, p138). Thus there does not exist $g \in C[0, 1]$ such that $1 = \|g\| = |L(g)|$, i.e., $\ker L$ is not w-proximinal.

Let A be a normed algebra but do not assume that A has an identity. We say that a left ideal I is modular if there exists $e \in A$ such that $A(1 - e) \equiv \{a - ae : a \in A\} \subset I$. e is called a right modular unit for I . Similarly, a right ideal I is modular if there exists a left modular unit for I . A two-sided ideal I is modular if it is modular both as a left and as a right ideal([3]).

DEFINITION 2.7. Let A be a normed algebra. A is said to have the first proximal property if every one-sided maximal modular ideal of A is w -proximal. A is said to have the second proximal property if every maximal modular ideal of a maximal commutative subalgebra of A is w -proximal.

By definition, a w -proximal subspace of a normed linear space has proper closure. So every w -proximal maximal modular left ideal is closed.

LEMMA 2.8. ([5]) Let M be a subspace of a normed linear space X . Given $x \in X$ with $d = d(x, M) > 0$, there exists an $f \in X'$ such that

$$\|f\| = 1, \quad f(M) = \{0\} \text{ and } f(x) = d(x, M).$$

Let $r(A)$ denotes the set of regular elements in a normed algebra A and $N(a, r)$ the open ball at $a \in A$ of radius r . For an element a of A , let $Sp_A(a)$, $Sp_A^L(a)$ ($Sp_A^R(a)$) denote the spectrum, left(right) spectrum of a with respect to A respectively.

THEOREM 2.9. Let A be a unital Banach algebra. Then every proper closed left or right ideal M of A is w -proximal. In particular, any unital Banach algebra has the first and second proximal properties.

Proof. Let M be any proper closed left or right ideal of A . If $x \in N(e, 1)$, then $\|e - x\| < 1$ and so x is regular in A by ([3], Lemma 2.1). Hence $e \in N(e, 1) \subseteq r(A) \subseteq M^c$ and $e \notin M$. Put $r = d(e, M)$. By Lemma 2.8, there exists $g \in A'$ such that $g(e) = r$, $\|g\| = 1$ and $g(M) = \{0\}$. Since e is an identity of A , $g(e) = 1$ and $\|e\| = 1$. Hence $1 = g(e) = r = d(e, M)$, i.e., M is w -proximal.

It is shown ([1]) that if A is a unital Banach algebra, then $Sp_A(a) \subset V(a)$. The following theorem gives the similar inclusion relation between the spectrum $Sp_A(a)$ and the relative numerical range of a in a non unital normed algebra A .

THEOREM 2.10. Let A be a normed algebra over \mathbb{C} having the first or second proximal property, and $a \in A$. Then $\{0\}^c \cap Sp_A^L(a) \subset V_x^L(A, a)$ for some $x \in A$ with $\|x\| = 1$.

Proof. Assume that A has the first proximal property. Let $\lambda \in Sp_A^L(a)$ and $\lambda \neq 0$. Define $j \equiv \lambda^{-1}a$. Then j is not left quasi-regular, i.e. there does not exist a $h \in A$ such that $h \circ j \equiv h + j - hj = 0$. Thus $A(1 - j)$ is a left modular ideal M of A , so $A(1 - j)$ is contained in a left maximal modular ideal of A . By the first proximal property, there exists $x \in A$ with $1 = \|x\| = d(x, M)$. By Lemma 2.8, there exists $f \in A^*$ such that $f(M) = \{0\}$, $1 = \|f\| = f(x)$. Also $x - xj \in M$ implies $0 = f(x - xj) = f(x) - f(xj) = 1 - \lambda^{-1}f(xa)$. Then $f(xa) = \lambda \in V_x^L(A, a)$

Assume that A has the second proximal property. For $a \in A$, a is an element of a maximal commutative subalgebra B of A . Let $\lambda \in \{0\}^c \cap Sp_A(a) = \{0\}^c \cap Sp_B(a)$. The second proximal property for A implies the first proximal property for B , so by the first argument, $\lambda \in V_x^L(B, a) \subseteq V_x^L(A, a)$ for some $x \in B$.

THEOREM 2.11. *Let A be a normed algebra over \mathbb{C} having the first or second proximal property and $a \in A$. Then $\{0\}^c \cap Sp_A^R(a) \subset V_x^R(A, a)$ for some $x \in A$ with $\|x\| = 1$.*

Proof. The proof is similar to that of Theorem 2.10.

COROLLARY 2.12. *Let A be a normed algebra over \mathbb{C} having the first or second proximal property and $a \in A$. Then $\{0\}^c \cap Sp_A(a) \subset V_x(A, a)$ for some $x \in A$ with $\|x\| = 1$.*

Proof. Since $Sp_A(a) = Sp_A^L(a) \cup Sp_A^R(a)$, for any $\lambda \in \{0\}^c \cap Sp_A(a)$, either $\lambda \in \{0\}^c \cap Sp_A^L(a)$ or $\lambda \in \{0\}^c \cap Sp_A^R(a)$. Assume that $\lambda \in \{0\}^c \cap Sp_A^L(a)$. Then by Theorem 2.10,

$$\lambda \in V_x^L(A, a) \subseteq V_x(A, a) \quad \text{for some } x \in A.$$

Similarly if $\lambda \in \{0\}^c \cap Sp_A^R(a)$, then by Theorem 2.11,

$$\lambda \in V_z^R(A, a) \subseteq V_z(A, a) \quad \text{for some } z \in A.$$

Therefore $\{0\}^c \cap Sp_A(a) \subset V_x(A, a)$ for some $x \in A$.

THEOREM 2.13. *Let A be a normed algebra over \mathbb{C} with the first or second proximal property and an identity e , but $\|e\|$ not necessarily 1. If $a \in A$ and $0 \in Sp_A^L(a)$, then $0 \in V_z^L(A, a)$ for some $z \in A$ with $\|z\| = 1$.*

Proof. If A has the first proximal property and $0 \in Sp_A^L(a)$, then a is not left regular, so Aa is a left modular ideal of A . Therefore Aa is contained in a left maximal modular ideal M of A . Note that since A has e , all ideals are modular. By hypothesis, there exists $z \in A$ with $1 = \|z\| = d(z, M)$. Also by Lemma 2.8, there exists $g \in A'$ with $1 = \|g\| = g(z)$ and $g(M) = \{0\}$. Thus $za \in M$, so $0 = g(za) \in V_z^L(A, a)$.

If A has the second proximal property, then a is an element of a maximal commutative subalgebra B of A , and $e \in B$. Since $0 \in Sp_A^L(a) = Sp_B^L(a)$ and the second proximal property for A implies the first proximal property for B , by the first argument, $0 \in V_z^L(B, a) = V_z^L(A, a)$ for some $z \in B$.

THEOREM 2.14. *Let A be a normed algebra over \mathbb{C} with the first or second proximal property and an identity e , but $\|e\|$ not necessarily 1. If $a \in A$ and $0 \in Sp_A^R(a)$, then $0 \in V_z^R(A, a)$ for some $z \in A$ with $\|z\| = 1$.*

Proof. The proof is similar to that of Theorem 2.13.

COROLLARY 2.15. *Let A be a normed algebra over \mathbb{C} with the first or second proximal property and an identity e , but $\|e\|$ not necessarily 1. If $a \in A$ and $0 \in Sp_A(a)$, then $0 \in V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.*

THEOREM 2.16. *Let A be a normed algebra such that all proper closed one-sided ideals of A are w -proximal. Let $a \in A$ with $0 \in Sp_A(a)$. If \overline{aA} or \overline{Aa} is properly contained in A , then $0 \in V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.*

Proof. Clearly \overline{aA} and \overline{Aa} are proper closed one-sided ideals of A . By hypothesis there exists $z \in A$ with $1 = \|z\| = d(z, \overline{Aa})$ or $d(z, \overline{aA})$. By Lemma 2.8, there exists $g \in A'$ such that $g(z) = \|g\| = 1$ and $g(\overline{aA})$ or $g(\overline{Aa}) = \{0\}$. So $0 = g(za) \in V_z^L(A, a)$ or $0 = g(az) \in V_z^R(A, a)$. Hence $0 \in V_z(A, a)$

THEOREM 2.17. *Let A be a finite dimensional complex normed algebra and let $a \in A$. Then*

- (1) $\{0\}^c \cap Sp_A(a) \subseteq V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.
- (2) aA and Aa are closed. If one of them is proper, then $0 \in V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.
- (3) If $aA = A = Aa$, then A has an identity e , a is invertible and $0 \notin Sp_A(a)$.
- (4) $Sp_A(a) \subseteq V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.

Proof. By Lemma 2.4, all proper ideals are closed and w-proximinal.

(1) By Corollary 2.12, $\{0\}^c \cap Sp_A(a) \subseteq V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.

(2) aA and Aa are clearly closed. If one of them is proper, then a is singular i.e., $0 \in Sp_A(a)$. By Theorem 2.16, $0 \in V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.

(3) There are $z, w \in A$ such that $za = a = aw$. Also $x \in A$ implies $x = pa = aq$ for some $p, q \in A$. So $xw = paw = pa = x$ and $zx = zaq = aq = x$. Hence $e = z = w$ and a is invertible.

(4) $(Aa) \cap (aA) \subset A$ or $= A$. If $(Aa) \cap (aA)$ is properly contained in A , then one of them is proper. By (1) and (2), $\{0\}^c \cap Sp_A(a) \subseteq V_z(A, a)$ and $0 \in V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$. Since $0 \in Sp_A(a)$, $Sp_A(a) \subseteq V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$.

If $(Aa) \cap (aA) = A$, then by (3) $0 \notin Sp_A(a)$ and by (1) $\{0\}^c \cap Sp_A(a) \subseteq V_z(A, a)$ for some $z \in A$ with $\|z\| = 1$. Hence $Sp_A(a) \subseteq V_z(A, a)$.

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