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ALMOST SURE CONVERGENCE FOR WEIGHTED SUMS OF I.I.D. RANDOM VARIABLES (II)

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1. Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of independent and identically distributed(i.i.d.) random variables with $EX = 0$ and $E|X|^p < \infty$ for some $p \geq 1$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants. The almost sure(a.s.) convergence of weighted sums $\sum_{i=1}^n a_{ni}X_i$ can be founded in Choi and Sung[1], Chow[2], Chow and Lai[3], Li *et al.*[4], Stout[6], Sung[8], Teicher[9], and Thrum[10]. As a special case of general statements, Teicher[9, p.341] obtained the following:
Let $\{X, X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX = 0$. If $\max_{1\leq i\leq n} |a_{ni}| = O(1/(n^{1/p} \log n))$ and $E|X|^p < \infty (1 \leq p \leq 2)$, then $\sum_{i=1}^n a_{ni}X_i$ converges to zero a.s.
Choi and Sung[1] and Sung[8]($p = 1$ and $1 < p < 2$, respectively) proved Teicher's result under the weaker condition $\max_{1\leq i\leq n} |a_{ni}| = O(1/(n^{1/p}(\log n)^{1-1/p}))$. The purpose of this paper is to weaken Teicher's condition $\max_{1\leq i\leq n} |a_{ni}| = O(1/(n^{1/p} \log n))$ for the case $p = 2$.

In what follows we will use the following notation: $\log x = \ln \max\{x, e\}$, where ln is the natural logarithm, and C denotes a positive constant which is not necessarily the same one in each appearance.

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2. Main result

The following two lemmas will be used in the proof of our main result.

LEMMA 1. *If* $EX^2 < \infty$ *then, for any* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} E|X| I(|X| > \epsilon \sqrt{\frac{n}{\log n}}) < \infty.$$

Proof. Noting that $\{n/\log n\}$ is an increasing sequence, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} E|X| I(|X| > \epsilon \sqrt{\frac{n}{\log n}}) \\
&= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n \log n}} \sum_{i=n}^{\infty} E|X| I(\epsilon \sqrt{\frac{i}{\log i}} < |X| \le \epsilon \sqrt{\frac{i+1}{\log(i+1)}}) \\
&= \sum_{i=1}^{\infty} E|X| I(\epsilon \sqrt{\frac{i}{\log i}} < |X| \le \epsilon \sqrt{\frac{i+1}{\log(i+1)}}) \sum_{n=1}^{i} \frac{1}{\sqrt{n \log n}} \\
&\le C \sum_{i=1}^{\infty} E|X| I(\epsilon \sqrt{\frac{i}{\log i}} < |X| \le \epsilon \sqrt{\frac{i+1}{\log(i+1)}}) \sqrt{\frac{i}{\log i}} \\
&\le C \sum_{i=1}^{\infty} P(\epsilon \sqrt{\frac{i}{\log i}} < |X| \le \epsilon \sqrt{\frac{i+1}{\log(i+1)}}) \frac{i}{\log i} \\
&\le C E X^2 < \infty,
\end{aligned}$$

since the first inequality follows from the following fact:

$$\sum_{n=1}^{i} \frac{1}{\sqrt{n \log n}} \le C \int_1^i \frac{1}{\sqrt{x \log x}}\, dx \le C \sqrt{\frac{i}{\log i}}.$$

The following lemma plays an essential role in our main result.

LEMMA 2. (Sung[7]). *Let* $\{X_{ni}, 1 \le i \le n, n \ge 1\}$ *be an array of rowwise independent random variables with* $EX_{ni} = 0$ *for* $1 \le i \le n$ *and* $n \ge 1$. *Set* $S_n = \sum_{i=1}^n X_{ni}$ *and* $s_n^2 = \sum_{i=1}^n EX_{ni}^2$. *Let* $\{k_n\}$ *be a sequence of positive constants such that* $k_n \to 0$ *as* $n \to \infty$. *Suppose that the following conditions hold:*

(i) $s_n^2 \le n$ *for* $n \ge 1$.
(ii) $|X_{ni}| \le k_n\sqrt{n}/\sqrt{\log n}$ a.s. *for* $1 \le i \le n$ *and* $n \ge 1$.

Then

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log n}} \le 1 \text{ a.s.}$$

Now we state and prove our main result.

THEOREM 3. *Let* $\{X, X_n, n \ge 1\}$ *be a sequence of i.i.d. random variables with* $EX = 0$ *and* $EX^2 = 1$. *Let* $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ *be a triangular array of constants satisfying*

$$\max_{1\le i\le n} |a_{ni}| \le \frac{1}{\sqrt{2n\log n}}. \tag{1}$$

Then

$$\limsup_{n\to\infty} \sum_{i=1}^n a_{ni}X_i \le 1 \quad \text{a.s.} \tag{2}$$

Proof. By Lemma 1 there exists a sequence $\{\epsilon_n\}$ of real numbers such that $0 < \epsilon_n \to 0$ and

$$\sum_{n=1}^\infty \frac{1}{\sqrt{n\log n}} E|X_n| I(|X_n| > \epsilon_n\sqrt{\frac{n}{\log n}}) < \infty. \tag{3}$$

Define $X_i' = X_i I(|X_i| \le \epsilon_i\sqrt{\frac{i}{\log i}})$, $X_i'' = X_i - X_i'$, and $X_{ni} = a_{ni}\sqrt{2n\log n}(X_i' - EX_i')$. Then we have by (1) that

$$\sum_{i=1}^n EX_{ni}^2 \le 2nEX^2 \log n \sum_{i=1}^n a_{ni}^2 \le n$$

and

$$\max_{1\le i\le n} |X_{ni}| \le 2 \max_{1\le i\le n} \epsilon_i \sqrt{\frac{i}{\log i}} = o(\sqrt{\frac{n}{\log n}}).$$

Hence, by Lemma 2, we have

$$\limsup_{n\to\infty} \sum_{i=1}^{n} a_{ni}(X_i' - EX_i') \le 1 \text{ a.s.}$$

To finish the proof, it is enough to show that

$$\sum_{i=1}^{n} a_{ni}(X_i'' - EX_i'') \to 0 \text{ a.s.} \tag{4}$$

By observing that

$$\begin{aligned}\max_{2^k\le n<2^{k+1}} |\sum_{i=1}^{n} a_{ni}(X_i'' - EX_i'')| &\le \frac{1}{\sqrt{2^{k+1}\log 2^k}} \sum_{i=1}^{2^{k+1}-1} (|X_i''| + E|X_i''|)\\ &\le \frac{C}{\sqrt{2^{k+1}\log 2^{k+1}}} \sum_{i=1}^{2^{k+1}} (|X_i''| + E|X_i''|),\end{aligned}$$

we will obtain (4) if we show that

$$\frac{1}{\sqrt{2^k\log 2^k}} \sum_{i=1}^{2^k} (|X_i''| + E|X_i''|) \to 0 \text{ a.s.} \tag{5}$$

as $k \to \infty$. From the Markov inequality and (3) we have that for any $\epsilon > 0$

$$\begin{aligned}&\sum_{k=1}^{\infty} P(\frac{1}{\sqrt{2^k\log 2^k}} \sum_{i=1}^{2^k} (|X_i''| + E|X_i''|) > \epsilon)\\ &\le \frac{2}{\epsilon} \sum_{k=1}^{\infty} \frac{1}{\sqrt{2^k\log 2^k}} \sum_{i=1}^{2^k} E|X_i''|\\ &= \frac{2}{\epsilon} \sum_{i=1}^{\infty} E|X_i''| \sum_{\{k:2^k\ge i\}} \frac{1}{\sqrt{2^k\log 2^k}}\\ &\le C \sum_{i=1}^{\infty} \frac{E|X_i''|}{\sqrt{i}\log i} < \infty,\end{aligned}$$

since the last inequality follows from the following:

$$\sum_{\{k:2^k\geq i\}} \frac{1}{\sqrt{2^k \log 2^k}} \leq \frac{1}{\sqrt{\log i}} \sum_{\{k:2^k\geq i\}} \frac{1}{\sqrt{2^k}} \leq \frac{1}{(1-1/\sqrt{2})\sqrt{i}\log i}.$$

Thus (5) holds by the Borel-Cantelli lemma, and so the proof is complete.

REMARK. In Theorem 3, if the condition (1) is replaced by the weaker condition

$$\max_{1\leq i\leq n} |a_{ni}| \leq \frac{1}{\sqrt{2n\log\log n}} \tag{6}$$

the result (2) can not hold. In fact, Li *et al.*[5] proved that for almost all choice of arrays satisfying (6)

$$\limsup_{n\to\infty} \sum_{i=1}^{n} a_{ni}X_i = \infty \quad \text{a.s.}$$

The following corollary shows that the right-hand side of (2) in Theorem 3 can be 0 if the condition (1) is replaced by the stronger condition (7).

COROLLARY 4. *Let $\{X, X_n, n\geq 1\}$ be a sequence of i.i.d. random variables with $EX=0$ and $EX^2<\infty$. Let $\{a_{ni}, 1\leq i\leq n, n\geq 1\}$ be a triangular array of constants satisfying*

$$\max_{1\leq i\leq n} |a_{ni}| = o(\frac{1}{\sqrt{n\log n}}). \tag{7}$$

Then

$$\sum_{i=1}^{n} a_{ni}X_i \to 0 \text{ a.s.}$$

Proof. Without loss of generality we assume $EX^2=1$. By the condition (7) there exists a sequence $\{\alpha_n\}$ of real numbers such that

$0 < \alpha_n \to 0$ and $\max_{1\le i\le n}|a_{ni}| \le \alpha_n/\sqrt{2n\log n}$. Then we have by Theorem 3 that

$$\limsup_{n\to\infty} \frac{\sum_{i=1}^n a_{ni}X_i}{\alpha_n} \le 1 \text{ a.s.}$$

From this result it follows that

$$\limsup_{n\to\infty} \sum_{i=1}^n a_{ni}X_i \le 0 \text{ a.s.}$$

By replacing X_i by $-X_i$ from the above statement we obtain

$$\liminf_{n\to\infty} \sum_{i=1}^n a_{ni}X_i \ge 0 \text{ a.s.}$$

Thus the conclusion follows.

REMARK. The condition (7) in Corollary 4 is weaker than Teicher's condition $\max_{1\le i\le n}|a_{ni}| = O(1/\sqrt{n}\log n)$.

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