Bull. Korean Math. Soc. **33** (1996), No. 3, pp. 411–417

THE DENJOY EXTENSION OF THE MCSHANE INTEGRAL

JAE MYUNG PARK AND DEOK HO LEE

1. Introduction

Some generalizations of the Riemann integral have been studied for real-valued functions. One of these generalizations leads to an integral, often called the McShane integral, that is equivalent to the Lebesgue integral.

We develop the extension of this integral for the case in which the function has values in a Banach space.

In [3], R.A.Gordon defined the Denjoy-Dunford, Denjoy-Pettis, and Denjoy-Bochner integrals of functions mapping an interval $[a, b]$ into a Banach space X, which are the extensions of Dunford, Pettis and Bochner integrals, respectively.

He showed that a Denjoy-Dunford(Denjoy-Bochner) integrable function on $[a, b]$ is Dunford(Bochner) integrable on some subinterval of $[a, b]$ and that for spaces that do not contain a copy of c_0, a Denjoy-Pettis integrable function on $[a, b]$ is Pettis integrable on some subinterval of $[a, b]$.

In this paper, we will study the Denjoy extension of the McShane integral, called the Denjoy-McShane integral.

2. Preliminaries

Throughout this paper, X will denote a real Banach space and X^* its dual.

Let $F : [a, b] \longrightarrow X$ be a function and let E be a subset of $[a, b]$.

Received October 31, 1995. Revised March 20, 1996.
1991 AMS Subject Classification: Primary 28B05; Secondary 46G10.
Key words and phrases: McShane integral, Denjoy-McShane integral.

The function F is AC(*absolutely continuous*) on E if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\Sigma_i ||F(d_i) - F(c_i)|| < \varepsilon$ whenever $\{[c_i, d_i]\}$ is a finite collection of nonoverlapping intervals that have endpoints in E and satisfy $\Sigma_i(d_i - c_i) < \delta$.

The function F is ACG(*generalized absolutely continuous*) on E if F is continuous on E and if E can be expressed as a countable union of sets on each of which F is AC.

We recall the following definitions.

DEFINITION 2.1. (a) The function $f : [a, b] \longrightarrow X$ is *Dunford integrable* on $[a, b]$ if x^*f is Lebesgue integrable on $[a, b]$ for each x^* in X^*. The *Dunford integral* of f on the measurable set $E \subset [a, b]$ is the vector x_E^{**} in X^{**} such that $x_E^{**}(x^*) = \int_E x^*f$ for all x^* in X^*.

(b) The function $f : [a, b] \longrightarrow X$ is *Pettis integrable* on $[a, b]$ if f is Dunford integrable on $[a, b]$ and x_E^{**} in X for every measurable set E in $[a, b]$.

(c) The function $f : [a, b] \longrightarrow X$ is *Bochner integrable* on $[a, b]$ if there exists an AC function $F : [a, b] \longrightarrow X$ such that F is differentiable almost everywhere on $[a, b]$ and $F' = f$ almost everywhere on $[a, b]$.

(d) A *McShane partition* of $[a, b]$ is a finite sequence $\langle([a_i, b_i], t_i)\rangle_{i \le n}$ such that $\langle[a_i, b_i]\rangle_{i \le n}$ is a non-overlapping family of intervals covering $[a, b]$ and $t_i \in [a, b]$ for each i. A *gauge* on $[a, b]$ is a function $\delta : [a, b] \longrightarrow (0, \infty)$. A McShane partition $\langle([a_i, b_i], t_i)\rangle_{i \le n}$ is *subordinate* to a gauge δ if $t_i - \delta(t_i) \le a_i \le b_i \le t_i + \delta(t_i)$ for every $i \le n$.

A function $\phi : [a, b] \longrightarrow X$ is *McShane integrable*, with McShane integral w, if for every $\varepsilon > 0$ there is a gauge $\delta : [a, b] \longrightarrow (0, \infty)$ such that

$$\left\| w - \sum_{i \le n} (b_i - a_i)\phi(t_i) \right\| < \varepsilon$$

for every McShane partition $\langle([a_i, b_i], t_i)\rangle_{i \le n}$ of $[a, b]$ subordinate to δ.

Let $F : [a, b] \longrightarrow X$ and let $t \in (a, b)$. A vector z in X is the *approximate derivative* of F at t if there exists a measurable set $E \subset$

$[a, b]$ that has t as a point of density such that

$$\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z$$

for the norm topology of X. In this case, we will write $F'_{ap}(t) = z$.

The function $f : [a, b] \longrightarrow R$ is *Denjoy integrable* on $[a, b]$ if there exists an ACG function $F : [a, b] \longrightarrow R$ such that $F'_{ap} = f$ almost everywhere on $[a, b]$. The function f is *Denjoy integrable on the set* $E \subset [a, b]$ if $f\chi_E$ is Denjoy integrable on $[a, b]$.

The following definitions can be found in [3].

DEFINITION 2.2. (a) The function $f : [a, b] \longrightarrow X$ is *Denjoy-Dunford integrable* on $[a, b]$ if for each $x^* \in X^*$ the function x^*f is Denjoy integrable on $[a, b]$ and if for every interval I in $[a, b]$ there exists a vector x_I^{**} in X^{**} such that $x_I^{**}(x^*) = \int_I x^*f$ for all x^* in X^*.

(b) The function $f : [a, b] \longrightarrow X$ is *Denjoy-Pettis integrable* on $[a, b]$ if f is Denjoy-Dunford integrable on $[a, b]$ and if $x_I^{**} \in X$ for every interval I in $[a, b]$.

(c) The function $f : [a, b] \longrightarrow X$ is *Denjoy-Bochner integrable* on $[a, b]$ if there exists an ACG function $F : [a, b] \longrightarrow X$ such that F is approximately differentiable almost everywhere on $[a, b]$ and $F'_{ap} = f$ almost everywhere on $[a, b]$.

3. The Denjoy extension of the McShane integral

In this section, we consider an extension of the McShane integral.

DEFINITION 3.1. The function $f : [a, b] \longrightarrow X$ is *Denjoy-McShane integrable* on $[a, b]$ if there exists a continuous function $F : [a, b] \longrightarrow X$ such that

(i) for each x^* in X^* x^*F is ACG and

(ii) for each x^* in X^* x^*F is approximately differentiable almost everywhere on $[a, b]$ and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a, b]$.

The following two theorems show that the Denjoy-McShane integral is the extension of the McShane integral and that it is also the extension of Denjoy-Bochner integral.

THEOREM 3.2. *If $f : [a,b] \longrightarrow X$ is McShane integrable on $[a,b]$, then f is Denjoy-McShane integrable.*

Proof. Suppose that $f : [a,b] \longrightarrow X$ is McShane integrable. Then for each x^* in X^* x^*f is McShane integrable and hence Lebesgue integrable.

Let $F(t) = (M)\int_a^t f$. Then F is continuous by [4, Theorem 8] and for each $x^* \in X^*$

$$x^*F(t) = (M)\int_a^t x^*f = (L)\int_a^t x^*f.$$

Hence x^*F is AC and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$. This implies that f is Denjoy-McShane integrable on $[a,b]$.

THEOREM 3.3. *If $f : [a,b] \longrightarrow X$ is Denjoy-Bochner integrable on $[a,b]$, then f is Denjoy-McShane integrable.*

Proof. Let $f : [a,b] \longrightarrow X$ be Denjoy-Bochner integrable. Then there exists an ACG function F such that $F'_{ap} = f$ almost everywhere on $[a,b]$.

It is easy to show that for each x^* in X^* x^*F is ACG and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$. Hence f is Denjoy-McShane integrable.

THEOREM 3.4. *If $f : [a,b] \longrightarrow X$ is Denjoy-McShane integrable on $[a,b]$, then f is Denjoy-Pettis integrable.*

Proof. Suppose that f is Denjoy-McShane integrable. Let $F(t) = (DM)\int_a^t f$. Since x^*F is ACG and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$ for each $x^* \in X^*$, x^*f is Denjoy integrable For every interval $[c,d]$ in $[a,b]$, we have

$$\begin{aligned} x^*(F(d) - F(c)) &= x^*F(d) - x^*F(c) \\ &= (D)\int_a^d x^*f - (D)\int_a^c x^*f = (D)\int_c^d x^*f. \end{aligned}$$

Since $F(d) - F(c) \in X$, f is Denjoy-Pettis integrable.

The following theorem was proved by R.A.Gordon. Its proof can be found in [3].

THEOREM 3.5. *Suppose that X contains no copy of c_0 and let $f : [a,b] \longrightarrow X$. If f is Denjoy-Pettis integrable on $[a,b]$, then every perfect set in $[a,b]$ contains a portion on which f is Pettis integrable.*

The next theorem is the analogue of Theorem 3.5 for Denjoy-McShane integrable functions.

THEOREM 3.6. *Suppose that X contains no copy of c_0 and let $f : [a,b] \longrightarrow X$ be measurable. If f is Denjoy-McShane integrable on $[a,b]$, then every perfect set in $[a,b]$ contains a portion on which f is McShane integrable.*

Proof. Let $f : [a,b] \longrightarrow X$ be Denjoy-McShane integrable on $[a,b]$. Since f is Denjoy-Pettis integrable by Theorem 3.4, it follows from Theorem 3.5 that every perfect set in $[a,b]$ contains a portion on which f is Pettis integrable. Since f is measurable, f is McShane integrable on that portion by [4, Theorem 17].

EXAMPLE 3.7. A Denjoy-McShane integrable function that is not Denjoy-Bochner integrable.

Let $\{r_k\}$ be a listing of the rational numbers in $[0,1]$ and for each pair of positive integers n and k let

$$I_n^k = \left(r_k + \frac{1}{n+1}, r_k + \frac{1}{n} \right).$$

For each k define $f_k : [0,1] \longrightarrow l_2$ by $f_k(t) = \{(n+1)\chi_{I_n^k}(t)\}$. Then the series $\Sigma_k 4^{-k} f_k$ is l_2-valued almost everywhere on $[0,1]$.

Let A be a set of measure zero in $[0,1]$ such that $\Sigma_k 4^{-k} f_k(t)$ is l_2-valued for all t in $[0,1] - A$.

Define $g : [0,1] \longrightarrow l_2$ by $g(t) = \Sigma_k 4^{-k} f_k(t)$ for t in $[0,1] - A$ and $g(t) = 0$ for t in A. It was proved in [3, Example 42] that g is measurable and Pettis integrable on $[0,1]$, but not Denjoy-Bochner integrable. By [4, Theorem 17], g is McShane integrable and hence Denjoy-McShane integrable by Theorem 3.2.

EXAMPLE 3.8. A Denjoy-Pettis integrable function that is not Denjoy- McShane integrable.

For each positive integer n let

$$I'_n = \left(\frac{1}{n+1}, \frac{n+\frac{1}{2}}{n(n+1)}\right), \quad I''_n = \left(\frac{n+\frac{1}{2}}{n(n+1)}, \frac{1}{n}\right)$$

and define $f_n : [0,1] \longrightarrow R$ by $f_n(t) = 2n(n+1)(\chi_{I'_n}(t) - \chi_{I''_n}(t))$. Then the sequence $\{f_n\}$ converges to 0 pointwise and it is not difficult to show that $\{\int_I f_n\}$ converges to 0 for each interval $I \subset [0,1]$. Define $f : [0,1] \longrightarrow c_0$ by $f(t) = \{f_n(t)\}$. Then it follows from [3, Example 44] that f is Dunford integrable and Denjoy-Pettis integrable on $[0,1]$.

To show that f is not Denjoy-McShane integrable, let $G(t) = (Dunford) - \int_a^t f$ and let $t_n = \frac{n+\frac{1}{2}}{n(n+1)}$ for each n. Then $t_n \to 0$ as $n \to \infty$, but $\int_0^{t_n} f_n = 1$ for each n. Hence $||G(t_n)||_{c_0} \geq 1$ and G is not continuous at 0. Suppose that f is Denjoy-McShane integrable. Then there exists a continuous function F such that x^*F is ACG and $(x^*F)'_{ap} = x^*f$ a.e. for each $x^* \in X^*$. By the Definition of the Denjoy integral, $x^*F(t) = (D)\int_0^t x^*f$. Since $x^*G(t) = (L)\int_0^t x^*f$ and every Lebesgue integrable function is Denjoy integrable, we have $x^*F(t) = x^*G(t)$ and $F = G$. This is a contradiction since F is continuous and G is not continuous on $[0,1]$.

References

1. D. H. Fremlin, *The Henstock and McShane integrals of vector-valued functions*, Illinois J. of Math. **38** (1994), 471-479.
2. D. H. Fremlin and J. Mendoza, *On the integration of vector-valued functions*, Illinois J. of Math. **38** (1994), 127-147.
3. R. A. Gordon, *The Denjoy extension of the Bochner, Pettis, and Dunford integrals*, Studia Math. **92** (1989), 73-91.
4. ———, *The McShane integral of Banach-valued functions*, Illinois J. Math. **34** (1990), 557-567.
5. ———, *Riemann integration in Banach spaces*, Rocky Mountain J. Math. **21** (1991), 923-949.
6. ———, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc. 1994.

Department of Mathematics, Chungnam National University, Taejon, 305-764, Korea

Department of Mathematics, Kongju National University, Kongju, 314-701, Korea