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REMARKS ON THE REIDEMEISTER NUMBERS

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0. Introduction

Let X be a connected compact polyhedron and $f : X \to X$ a selfmap of X. The Reidemeister number of f is denoted by $R(f)$.

A lower bound of the Reidemeister number has been obtained in [1] as follows:

$$|\mathrm{Coker}(1 - f_{1*})| \leq R(f),$$

where $f_{1*} : H_1(X) \to H_1(X)$ is the homomorphism induced by f and $H_1(X)$ is the 1-dimensional homology group of X.

In this paper, we obtain an upper bound of the Reidemeister number $R(f)$ as follows:

$$R(f) \leq |\mathrm{Coker}(1 - f_{1*})||D(\pi_1(X, x_0))|,$$

where $D(\pi_1(X, x_0))$ is the commutator subgroup of the fundamental group $\pi_1(X, x_0)$.

Thus we have

$$|\mathrm{Coker}(1 - f_{1*})| \leq R(f) \leq |\mathrm{Coker}(1 - f_{1*})||D(\pi_1(X, x_0))|.$$

Obviously, it follows that if $\pi_1(X, x_0)$ is abelian then $R(f) = |\mathrm{Coker}(1 - f_{1*})|$.

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1. Preliminaries

Let X be a connected compact polyhedron and $f : X \to X$ a selfmap of X. The fundamental group $\pi_1(X, x_0)$ is simply denoted by π. Since there exists always a map $g : X \to X$ such that $f \simeq g : X \to X$ and $g(x_0) = x_0$, we may suppose $f(x_0) = x_0$.

Let $H \lhd \pi$ and $f_\pi(H) \subset H$, where $H \lhd \pi$ denotes that H is normal subgroup of π and $f_\pi : \pi \to \pi, \langle a \rangle \mapsto \langle f \circ a \rangle$.

Up to isomorphism there exists a unique regular covering space $({}_H\tilde{X}, {}_HP)$ of X such that

$$({}_HP)_\pi(\pi_1({}_H\tilde{X}, {}_H\tilde{x}_0)) = H, \quad \text{where} \quad {}_H\tilde{x}_0 \in {}_HP^{-1}(x_0).$$

We construct $({}_H\tilde{X}, {}_HP)$ as follows:

$$\begin{aligned} {}_H\tilde{X} &= \{H\langle c \rangle : \langle c \rangle \quad \text{is the path class in } X \\ &\qquad\qquad\qquad \text{with the initial point } x_0\} \\ {}_HP(H\langle c \rangle) &= c(1) \end{aligned}$$

The base point of ${}_H\tilde{X}$ is picked ${}_H\tilde{x}_0 = H\langle e_0 \rangle$, where e_0 is the point path at x_0 and $\langle e_0 \rangle = e$ is the identity element of π.

The set of all liftings of $f : X \to X$ on ${}_H\tilde{X}$ is denoted by $\mathrm{Lift}(f, H)$. Especially, the set $\mathcal{D}({}_H\tilde{X}, {}_HP) := \mathrm{Lift}(i_X, H)$ forms a group under the composition of maps, where i_X is the identity map of X.

The correspondence

$$\begin{aligned} \lambda_H : \mathcal{D}({}_H\tilde{X}, {}_HP) &\to \pi/H \\ \ell_{[\alpha]} &\longmapsto [\alpha] := H\alpha \end{aligned}$$

is an isomorphism, where $\ell_{[\alpha]}(H\langle e_0 \rangle) = [\alpha]$. Hence we will identify $\ell_{[\alpha]}$ with $[\alpha]$ and $\mathcal{D}({}_H\tilde{X}, {}_HP)$ with π/H.

If ${}_H\tilde{f}, {}_H\tilde{f}' \in \mathrm{Lift}(f, H)$ then there exists a unique $[\alpha] \in \pi/H$ such that ${}_H\tilde{f}' = [\alpha] \circ {}_H\tilde{f}$. Hence for some ${}_H\tilde{f} \in \mathrm{Lift}(f, H)$ we have

$$\mathrm{Lift}(f, H) = \{[\alpha] \circ {}_H\tilde{f} : [\alpha] \in \pi/H\}$$

Two liftings ${}_H\tilde{f}, {}_H\tilde{f}' \in \text{Lift}(f, H)$ are said to be *conjugate* if there exists $[\alpha] \in \pi/H$ such that ${}_H\tilde{f}' = [\alpha] \circ {}_H\tilde{f} \circ [\alpha]^{-1}$. The equivalence classes by conjugacy are called *lifting classes* and the lifting class of ${}_H\tilde{f}$ is denoted by

$$[{}_H\tilde{f}] = \{[\alpha] \circ {}_H\tilde{f} \circ [\alpha]^{-1} : [\alpha] \in \pi/H\}$$

The set of all lifting classes is denoted by $\text{Lift}'(f, H)$, and $R(f, H) := |\text{Lift}'(f, H)|$ is called the H-Reidemeister number of f.

For every $[\alpha] \in \pi/H$ and some ${}_H\tilde{f} \in \text{Lift}(f, H)$, the composition ${}_H\tilde{f} \circ [\alpha]$ is also a lifting of f on ${}_H\tilde{X}$, so there exists a unique element $[\alpha'] \in \pi/H$ such that $[\alpha'] \circ {}_H\tilde{f} = {}_H\tilde{f} \circ [\alpha]$. This correspondence

$$\begin{aligned} {}_H\tilde{f}_\pi : \pi/H &\to \pi/H \\ [\alpha] &\longmapsto [\alpha'] \end{aligned}$$

is determined by ${}_H\tilde{f}$ and is a homomorphism. If π/H is a commutative group then ${}_H\tilde{f}\pi$ does not depend on the choice of ${}_H\tilde{f} \in \text{Lift}(f, H)$.

For $[\alpha], [\alpha'] \in \pi/H$ and ${}_H\tilde{f} \in \text{Lift}(f, H), [[\alpha] \circ {}_H\tilde{f}] = [[\alpha'] \circ {}_H\tilde{f}]$ if and only if there exists a $[\gamma] \in \pi/H$ such that

$$[\alpha'] = [\gamma][\alpha]{}_H\tilde{f}_\pi([\gamma]^{-1}).$$

We introduce the following notations:

$$\begin{aligned} T([\alpha], {}_H\tilde{f}) &:= \{[\gamma][\alpha]{}_H\tilde{f}_\pi([\gamma]^{-1}) : [\gamma] \in \pi/H\}, \\ (\pi/H)' &:= \{T([\alpha], {}_H\tilde{f}) : [\alpha] \in \pi/H\}. \end{aligned}$$

Obviously, $R(f, H) = |(\pi/H)'|$.

If π/H is a commutative group then $T([\alpha], {}_H\tilde{f})$ does not depend on the choice of ${}_H\tilde{f} \in \text{Lift}(f, H)$. In this case $T([\alpha], {}_H\tilde{f})$ is simply denoted by $T([\alpha], H)$.

If $H = \{e\}$ then the notations $({}_H\tilde{X}, {}_HP)$, $\text{Lift}(f, H)$, ${}_H\tilde{f}$, $\mathcal{D}({}_H\tilde{X}, {}_HP)$, $\text{Lift}'(f, H)$, $R(f, H)$, λ_H, ${}_H\tilde{f}_\pi$, $T([\alpha], {}_H\tilde{f})$, $T([\alpha], H)$, and $(\pi/H)'$ are simply denoted by $(\tilde{X}, P)$, $\text{Lift}(f)$, $\tilde{f}$, $\mathcal{D}(\tilde{X}, P)$, $\text{Lift}'(f)$, $R(f)$, λ, $\tilde{f}_\pi$, $T(\alpha, \tilde{f})$, $T(\alpha)$, and π' respectively.

Here $(\tilde{X}, P)$ is a universal covering space of X and $R(f)$ is called the Reidemeister number of f.

The covering space homomorphism

$$\begin{aligned} \varphi : (\tilde{X}, P) &\to ({}_H\tilde{X}, {}_HP) \\ \langle c \rangle &\longmapsto H\langle c \rangle \end{aligned}$$

induces a surjection

$$\begin{aligned} \varphi' : \mathrm{Lift}(f) &\to \mathrm{Lift}(f, H) \\ \tilde{f} &\longmapsto \varphi'(\tilde{f}), \quad \text{where } \varphi'(\tilde{f}) \circ \varphi = \varphi \circ \tilde{f}. \end{aligned}$$

Especially, if $f = i_X$ then we have

$$\begin{aligned} \varphi' : \mathcal{D}(\tilde{X}, P) = \pi &\to \mathcal{D}({}_H\tilde{X}, {}_HP) = \pi/H \\ \alpha &\longmapsto \varphi'(\alpha). \end{aligned}$$

where $\varphi'(\alpha) \circ \varphi = \varphi \circ \alpha$.

Note that

$$\begin{aligned} &(\varphi \circ \alpha)(\langle e_0 \rangle) = \varphi(\alpha(\langle e_0 \rangle)) = \varphi(\ell_\alpha(\langle e_0 \rangle)) = \varphi(\alpha) = H\alpha = [\alpha], \\ &(\varphi'(\alpha) \circ \varphi)(\langle e_0 \rangle) = \varphi'(\alpha)(\varphi(\langle e_0 \rangle)) = \varphi'(\alpha)(H\langle e_0 \rangle). \end{aligned}$$

Hence $\varphi'(\alpha)(H\langle e_0 \rangle) = [\alpha] = \ell_{[\alpha]}(H\langle e_0 \rangle)$. It follows that

$$\varphi'(\alpha) = \ell_{[\alpha]} = [\alpha].$$

Thus we have

$$[\alpha] \circ \varphi = \varphi \circ \alpha$$

2. An upper bound of the Reidemeister number $R(f)$

LEMMA 1. *If π/H is commutative group, then*

$$R(f,H) = |(\pi/H)/T([e],H)|.$$

Proof. Since π/H is a commutative group, we have

$$\begin{aligned} T([\alpha],H) =& \{[\gamma][\alpha]_H\tilde{f}_\pi([\gamma]^{-1}) : [\gamma] \in \pi/H\} \\ =& [\alpha]\{[\gamma]_H\tilde{f}_\pi([\gamma]^{-1}) : [\gamma] \in \pi/H\} \\ =& [\alpha]T([e],H). \end{aligned}$$

Moreover, it is easy to see that $T([e],H)$ is a normal subgroup of π/H. Hence we have

$$\begin{aligned} R(f,H) =& |(\pi/H)'| \\ =& |\{[\alpha]T([e],H) : [\alpha] \in \pi/H\}| \\ =& |(\pi/H)/T([e],H)|. \qquad \square \end{aligned}$$

LEMMA 2. *If $\tilde{f} \in Lift(f)$, ${}_H\tilde{f} \in Lift(f,H)$ and ${}_H\tilde{f} = \varphi'(\tilde{f})$. Then*

$${}_H\tilde{f}_\pi([\gamma]) = [\tilde{f}_\pi(\gamma)].$$

Proof. By the definiton of ${}_H\tilde{f}_\pi$, we have

$${}_H\tilde{f}_\pi([\gamma]) \circ {}_H\tilde{f} = {}_H\tilde{f} \circ [\gamma].$$

Observe that

$$\begin{aligned} {}_H\tilde{f}_\pi([\gamma]) \circ {}_H\tilde{f} \circ \varphi =& {}_H\tilde{f} \circ [\gamma] \circ \varphi = {}_H\tilde{f} \circ \varphi \circ \gamma = \varphi \circ \tilde{f} \circ \gamma \\ =& \varphi \circ \tilde{f}_\pi(\gamma) \circ \tilde{f} = [\tilde{f}_\pi(\gamma)] \circ \varphi \circ \tilde{f} = [\tilde{f}_\pi(\gamma)] \circ {}_H\tilde{f} \circ \varphi. \end{aligned}$$

Since φ is a surjection, if follows that

$${}_H\tilde{f}_\pi([\gamma]) \circ {}_H\tilde{f} = [\tilde{f}_\pi(\gamma)] \circ {}_H\tilde{f}.$$

So that

$${}_H\tilde{f}_\pi([\gamma]) = [\tilde{f}_\pi(\gamma)]. \qquad \square$$

Note that $\varphi' :$ Lift$(f) \to$ Lift(f,H) induces a surjection

$$\begin{aligned} \overline{\varphi'} : \text{Lift}'(f) &\to \text{Lift}'(f,H) \\ [\tilde{f}] &\longmapsto [\varphi'(\tilde{f})]. \end{aligned}$$

Lemma 3. *If $_H\tilde{f} \in Lift(f, H), \tilde{f} \in Lift(f)$ and $\varphi'(\tilde{f}) = {_H\tilde{f}}$. Then*

$$(\overline{\varphi'})^{-1}([_H\tilde{f}]) = \{[\alpha \circ \tilde{f}] \in Lift'(f) : \alpha \in HT(e, \tilde{f})\}.$$

Proof. Observe that

$$\begin{aligned}
[\alpha \circ \tilde{f}] \in (\overline{\varphi'})^{-1}([_H\tilde{f}]) &\iff \overline{\varphi'}([\alpha \circ \tilde{f}]) = [_H\tilde{f}] \\
&\iff [\varphi'(\alpha \circ \tilde{f})] = [_H\tilde{f}] \\
&\iff \exists\ \gamma \in \pi,\ \varphi'(\alpha \circ \tilde{f}) = [\gamma] \circ {_H\tilde{f}} \circ [\gamma]^{-1} \\
&\iff \exists\ \gamma \in \pi,\ \varphi \circ (\alpha \circ \tilde{f}) = ([\gamma] \circ {_H\tilde{f}} \circ [\gamma]^{-1}) \circ \varphi.
\end{aligned}$$

Note that

$$\begin{aligned}
\varphi \circ (\alpha \circ \tilde{f}) =&(\varphi \circ \alpha) \circ \tilde{f} = ([\alpha] \circ \varphi) \circ \tilde{f} = [\alpha] \circ (\varphi \circ \tilde{f}) \\
=&[\alpha] \circ ({_H\tilde{f}} \circ \varphi) = ([\alpha] \circ {_H\tilde{f}}) \circ \varphi
\end{aligned}$$

and

$$\begin{aligned}
([\gamma] \circ {_H\tilde{f}} \circ [\gamma]^{-1}) \circ \varphi =&([\gamma]\, {_H\tilde{f}_\pi}([\gamma]^{-1}) \circ {_H\tilde{f}}) \circ \varphi \\
=&([\gamma][\tilde{f}_\pi(\gamma^{-1})] \circ {_H\tilde{f}}) \circ \varphi = ([\gamma \tilde{f}_\pi(\gamma^{-1})] \circ {_H\tilde{f}}) \circ \varphi.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&[\alpha \circ \tilde{f}] \in (\overline{\varphi'})^{-1}([_H\tilde{f}]) \\
&\qquad \iff \exists\ \gamma \in \pi, ([\alpha] \circ {_H\tilde{f}}) \circ \varphi = ([\gamma \tilde{f}_\pi(\gamma^{-1})] \circ {_H\tilde{f}}) \circ \varphi \\
&\qquad \iff \exists\ \gamma \in \pi, [\alpha] \circ {_H\tilde{f}} = [\gamma \tilde{f}_\pi(\gamma^{-1})] \circ {_H\tilde{f}} \\
&\qquad \iff \exists\ \gamma \in \pi, [\alpha] = [\gamma \tilde{f}_\pi(\gamma^{-1})] \\
&\qquad \iff \exists\ \gamma \in \pi, \alpha \in H\gamma \tilde{f}_\pi(\gamma^{-1}) \\
&\qquad \iff \alpha \in HT(e, \tilde{f}).
\end{aligned}$$

Therefore $(\overline{\varphi'})^{-1}([_H\tilde{f}]) = \{[\alpha \circ \tilde{f}] \in \text{Lift}'(f) : \alpha \in HT(e, \tilde{f})\}$. □

LEMMA 4. *For any* ${}_H\tilde{f} \in Lift(f, H)$, *we have*

$$|(\overline{\varphi'})^{-1}([{}_H\tilde{f}])| \leq |H|.$$

Proof. Let $\alpha \in HT(e, \tilde{f})$. Then there exist $h_\alpha \in H, u_\alpha \in T(e, \tilde{f})$ such that $\alpha = h_\alpha u_\alpha$. Since $u_\alpha \in T(e, \tilde{f}) = \{t\tilde{f}_\pi(t^{-1}) : t \in \pi\}$, there exists $t_\alpha \in \pi$ such that $u_\alpha = t_\alpha \tilde{f}_\pi(t_\alpha^{-1})$. Hence $\alpha = h_\alpha t_\alpha \tilde{f}_\pi(t_\alpha^{-1})$. Thus for any $\gamma \in \pi$ and $\alpha \in HT(e, \tilde{f})$, we have

$$\gamma\alpha\tilde{f}_\pi(\gamma^{-1}) = \gamma h_\alpha t_\alpha \tilde{f}_\pi(t_\alpha^{-1})\tilde{f}_\pi(\gamma^{-1}) = \gamma h_\alpha t_\alpha \tilde{f}_\pi((\gamma t_\alpha)^{-1}).$$

Since $\gamma h_\alpha \in \gamma H = H\gamma$, there is $h_\alpha(\gamma) \in H$ such that $\gamma h_\alpha = h_\alpha(\gamma)\gamma$. Thus $\gamma\alpha\tilde{f}_\pi(\gamma^{-1}) = h_\alpha(\gamma)\gamma t_\alpha \tilde{f}_\pi((\gamma t_\alpha)^{-1}) \in HT(e, \tilde{f})$. Hence the correspondence

$$g : \pi \times HT(e, \tilde{f}) \to HT(e, \tilde{f})$$
$$(\gamma, \alpha) \longmapsto \gamma\alpha\tilde{f}_\pi(\gamma^{-1})$$

is well-defined. It is easy to see that g is a left action of π on $HT(e, \tilde{f})$.

For any $\alpha \in HT(e, \tilde{f})$, consider the orbit of α

$$\Omega_\alpha = \{\gamma\alpha\tilde{f}_\pi(\gamma^{-1}) : \gamma \in \pi\}.$$

Note that $[\alpha \circ \tilde{f}] = [\alpha' \circ \tilde{f}]$ if and only if there exists $\gamma \in \pi$ such that $\alpha' = \gamma\alpha\tilde{f}_\pi(\gamma^{-1})$. Hence

$$|(\overline{\varphi'})^{-1}([{}_H\tilde{f}])| = |\{\Omega_\alpha : \alpha \in HT(e, \tilde{f})\}|.$$

By the proceding argument, we have

$$\gamma\alpha\tilde{f}_\pi(\gamma^{-1}) = h_\alpha(\gamma)\gamma t_\alpha \tilde{f}_\pi((\gamma t_\alpha)^{-1}), \text{ where } \alpha = h_\alpha t_\alpha \tilde{f}_\pi(t_\alpha^{-1})$$

and $h_\alpha(\gamma)\gamma = \gamma h_\alpha, h_\alpha(\gamma) \in H$. Hence

$$\Omega_\alpha = \{h_\alpha(\gamma)\gamma t_\alpha \tilde{f}_\pi((\gamma t_\alpha)^{-1}) : \gamma \in \pi\}.$$

It follows that if $\gamma = t_\alpha^{-1}$ then $h_\alpha(t_\alpha^{-1}) \in \Omega_\alpha$. Thus for every $\alpha \in HT(e, \tilde{f})$ there exists $h_\alpha(t_\alpha^{-1}) \in H$ such that

$$\Omega_\alpha = \Omega_{h_\alpha(t_\alpha^{-1})}.$$

Therefore

$$|\{\Omega_\alpha | \alpha \in HT(e, \tilde{f})\}| = |\{\Omega_{h_\alpha(t_\alpha^{-1})} | h_\alpha(t_\alpha^{-1}) \in H\}| \leq |H|.$$

So that for any ${}_H\tilde{f} \in \mathrm{Lift}(f, H)$, we have

$$|(\overline{\varphi'})^{-1}([{}_H\tilde{f}])| \leq |H|. \qquad \square$$

LEMMA 5. *If $D(\pi) \subset H$ then $HT(e, \tilde{f})$ is a normal subgroup of π, where $D(\pi)$ is the commutator subgroup of π.*

Proof. Obviously, $HT(e, \tilde{f}) \neq \emptyset$. Let $t_1, t_2 \in HT(e, \tilde{f})$. Then there exist $h_1, h_2 \in H$ and $\gamma_1, \gamma_2 \in \pi$ such that $t_1 = h_1\gamma_1\tilde{f}_\pi(\gamma_1^{-1}), t_2 = h_2\gamma_2\tilde{f}_\pi(\gamma_2^{-1})$. Observe that

$$\begin{aligned} t_1t_2^{-1} =& h_1\gamma_1\tilde{f}_\pi(\gamma_1^{-1})\tilde{f}_\pi(\gamma_2)\gamma_2^{-1}h_2^{-1} = h_1\gamma_1\tilde{f}_\pi(\gamma_2^{-1}\gamma_2)\gamma_2^{-1}h_2^{-1} \\ =&[h_1\gamma_1\tilde{f}_\pi(\gamma_1^{-1}\gamma_2)\gamma_2^{-1}, h_2^{-1}]h_2^{-1}h_1\gamma_1\tilde{f}_\pi(\gamma_1^{-1}\gamma_2)\gamma_2^{-1} \\ =&[h_1\gamma_1\tilde{f}_\pi(\gamma_1^{-1}\gamma_2)\gamma_2^{-1}, h_2^{-1}]h_2^{-1}h_1[\gamma_1, \tilde{f}_\pi(\gamma_1^{-1}\gamma_2)\gamma_2^{-1}]\tilde{f}_\pi(\gamma_1^{-1}\gamma_2)\gamma_2^{-1}\gamma_1 \\ =&[h_1\gamma_1\tilde{f}_\pi(\gamma_1^{-1}\gamma_2)\gamma_2^{-1}, h_2^{-1}]h_2^{-1}h_1[\gamma_1, \tilde{f}_\pi(\gamma_1^{-1}\gamma_2)\gamma_2^{-1}] \\ &[\tilde{f}_\pi(\gamma_1^{-1}\gamma_2), \gamma_2^{-1}\gamma_1]\gamma_2^{-1}\gamma_1 \cdot \tilde{f}_\pi((\gamma_2^{-1}\gamma_1)^{-1}) \in HT(e, \tilde{f}). \end{aligned}$$

Hence $HT(e, \tilde{f})$ is a subgroup of π.

For $\alpha \in HT(e, \tilde{f})$ and $t \in \pi$, we have

$$\begin{aligned} t\alpha t^{-1} =& th\gamma\tilde{f}_\pi(\gamma^{-1})t^{-1}, \text{ where } \alpha = h\gamma\tilde{f}_\pi(\gamma^{-1}), h \in H \\ =& h't\gamma\tilde{f}_\pi(\gamma^{-1})t^{-1}, \text{ where } h't = th, h' \in H \\ =& h'[t\gamma\tilde{f}_\pi(\gamma^{-1}), t^{-1}]t^{-1}t\gamma\tilde{f}_\pi(\gamma^{-1}) \\ =& h'[t\gamma\tilde{f}_\pi(\gamma^{-1}), t^{-1}]\gamma\tilde{f}_\pi(\gamma^{-1}) \in HT(e, \tilde{f}). \end{aligned}$$

Hence $HT(e, \tilde{f})$ is a normal subgroup of π. $\square$

THEOREM 1. *It* $D(\pi) \subset H$ *then*

$$R(f) \leq |H||\pi/HT(e,\tilde{f})|.$$

Proof. Since $D(\pi) \subset H, \pi/H$ is a commutative group. Note that

$$\begin{aligned} R(f) =&|\mathrm{Lift}'(f)| = |\bigcup\{(\overline{\varphi'})^{-1}([_H\tilde{f}]) : [_H\tilde{f}] \in \mathrm{Lift}'(f,H)\}| \\ =&\sum |(\overline{\varphi'})^{-1}([_H\tilde{f}])|[_H\tilde{f}] \in \mathrm{Lift}'(f,H) \end{aligned}$$

It follows from Lemma 4 and Lemma 1 that

$$R(f) \leq |H||\mathrm{Lift}'(f,H)| = |H|R(f,H) = |H||(\pi/H)/T([e],H)|.$$

Note that

$$\begin{aligned} T([e],H) =&\{[\gamma]_H\tilde{f}_\pi([\gamma]^{-1}) : [\gamma] \in \pi/H\} \\ =&\{\gamma\tilde{f}_\pi(\gamma^{-1})] : \gamma \in \pi\} \\ =&\{H\gamma\tilde{f}_\pi(\gamma^{-1}) : \gamma \in \pi\} = HT(e,\tilde{f})/H. \end{aligned}$$

Hence we have

$$R(f) \leq |H||(\pi/H)/(HT(e,\tilde{f})/H)| = |H||\pi/HT(e,\tilde{f})|. \qquad \square$$

THEOREM 2. $R(f) \leq |D(\pi)||\pi/D(\pi)T(e,\tilde{f})|$ *and for* $M := \{H \lhd \pi : D(\pi) \subset H, f_\pi(H) \subset H\}$, *we have*

$$|D(\pi)||\pi/D(\pi)T(e,\tilde{f})| = \min\{|H||\pi/HT(e,\tilde{f})| : H \in M\}.$$

Proof. Note that $D(\pi) \lhd \pi$ and $f_\pi(D(\pi)) \subset D(\pi)$. From Theorem 1, we have

$$R(f) \leq |D(\pi)||\pi/D(\pi)T(e,\tilde{f})|.$$

Now let us show that if $H_1, H_2 \in M$ and $H_1 \subset H_2$, then

$$|H_1||\pi/H_1T(e,\tilde{f})| \leq |H_2||\pi/H_2T(e,\tilde{f})|.$$

Consider the homomorphism

$$\psi : H_2/H_1 \to H_2T(e,\tilde{f})/H_1T(e,\tilde{f})$$
$$H_1 a \longmapsto H_1T(e,\tilde{f})a.$$

If $H_1T(e,\tilde{f})b \in H_2T(e,\tilde{f})/H_1T(e,\tilde{f})$, then $b \in H_2T(e,\tilde{f})$. Hence there exist $h \in H_2, t \in T(e,\tilde{f})$ such that $b = ht = [h,t]th$. Thus

$$H_1T(e,\tilde{f})b = H_1T(e,\tilde{f})[h,t]th = H_1T(e,\tilde{f})h = \psi(H_1h).$$

Therfore ψ is an epimorphism. It follows that

$$|H_2T(e,\tilde{f})/H_1T(e,\tilde{f})| \le |H_2/H_1|.$$

So that we have

$$|H_1||H_2T(e,\tilde{f})/H_1T(e,\tilde{f})||\pi/H_2T(e,\tilde{f})| \le |H_1||H_2/H_1||\pi/H_2T(e,\tilde{f})|;$$

that is,

$$|H_1||\pi/H_1T(e,\tilde{f})| \le |H_2||\pi/H_2T(e,\tilde{f})|.$$

Note that $D(\pi) \in M$ and for any $H \in M, D(\pi) \subset H$. Hence for any $H \in M, |D(\pi)||\pi/D(\pi)T(e,\tilde{f})| \le |H||\pi/T(e,\tilde{f})|$. □

3. An estimation of the Reidemeister numbers

THEROEM 3. $|\pi/D(\pi)T(e,f)| \le R(f) \le |D(\pi)||\pi/D(\pi)T(e,f)|$, where $T(e,f) = \{\gamma f_\pi(\gamma^{-1}) : \gamma \in \pi\}$.

Proof. By Theorem 2, we have

$$R(f) \le |D(\pi)|\left|\pi/D(\pi)T(e,\tilde{f})\right|.$$

Moreover, it is well known that

$$|\mathrm{Coker}(1 - f_{1*})| \le R(f),$$

where $f_{1*} : H_1(X) \to H_1(X)$ is the homomorphism induced by f and $H_1(X)$ the 1-dimensional homology group of X[1]. Hence it is sufficient to show that

$$D(\pi)T(e, f) = D(\pi)T(e, \tilde{f})$$

and

$$|\mathrm{Coker}(1 - f_{1*})| = |\pi / D(\pi)T(e, \tilde{f})|.$$

Consider the commutative diagram

$$\begin{array}{ccc} \pi & \xrightarrow{\tilde{f}_\pi} & \pi \\ \theta \downarrow & & \theta \downarrow \\ H_1(X) & \xrightarrow{f_{1*}} & H_1(X) \end{array}$$

and the composition $\eta \circ \theta$

$$\pi \xrightarrow{\theta} H_1(X) \xrightarrow{\eta} \mathrm{Coker}(H_1(X) \xrightarrow{1 - f_{1*}} H_1(X)),$$

where θ is the abelianization ; i.e., θ is a surjective homomorphism, ker $\theta = D(\pi)$, and η is the natural projection. Since $\eta \circ \theta$ is a surjective homomorphism, we have

$$\mathrm{Coker}(1 - f_{1*}) \cong \pi / \ker(\eta \circ \theta).$$

Observe that

$$\begin{aligned} \alpha \in \ker(\eta \circ \theta) &\iff \eta \circ \theta(\alpha) = 0 \\ &\iff \theta(\alpha) \in (1 - f_{1*})(H_1(X)) = (1 - f_{1*})(\theta(\pi)) \\ &\iff \exists\ \gamma \in \pi, \theta(\alpha) = (1 - f_{1*})(\theta(\gamma)) = \theta(\gamma) - f_{1*} \circ \theta(\gamma) \\ &\qquad = \theta(\gamma) - \theta \circ \tilde{f}_\pi(\gamma) = \theta(\gamma \tilde{f}_\pi(\gamma^{-1})) \\ &\iff \exists\ \gamma \in \pi, \alpha(\gamma \tilde{f}_\pi(\gamma^{-1}))^{-1} \in \ker\theta = D(\pi) \\ &\iff \exists\ \gamma \in \pi, \alpha \in D(\pi)\gamma \tilde{f}_\pi(\gamma^{-1}) \\ &\iff \alpha \in D(\pi)T(e, \tilde{f}). \end{aligned}$$

Hence we have

$$\ker(\eta \circ \theta) = D(\pi)T(e, \tilde{f})$$

and

$$\mathrm{Coker}(1 - f_{1*}) \cong \pi / D(\pi)T(e, \tilde{f}).$$

It follows that

$$|\mathrm{Coker}(1 - f_{1*})| = |\pi / D(\pi)T(e, \tilde{f})|.$$

It remains to show that

$$D(\pi)T(e, \tilde{f}) = D(\pi)T(e, f).$$

Let $\tilde{f}$ be the lifting of f such that $\tilde{f}(\langle e_0 \rangle) = \beta$. Then for any $\alpha \in \pi$, we have

$$\tilde{f}_\pi(\alpha) = \beta f_\pi(\alpha)\beta^{-1} \text{ or } f_\pi(\alpha) = \beta^{-1}\tilde{f}_\pi(\alpha)\beta.$$

If $u \in D(\pi)T(e, \tilde{f})$ then there exist $d \in D(\pi)$ and $\gamma \in \pi$ such that

$$\begin{aligned} u =& d\gamma\tilde{f}_\pi(\gamma^{-1}) = d\gamma\beta f_\pi(\gamma^{-1})\beta^{-1} \\ =& d[\gamma, \beta]\beta\gamma f_\pi(\gamma^{-1})\beta^{-1} \\ =& d[\gamma, \beta][\beta\gamma f_\pi(\gamma^{-1}), \beta^{-1}]\beta^{-1}\beta\gamma f_\pi(\gamma^{-1}) \\ =& d[\gamma, \beta][\beta\gamma f_\pi(\gamma^{-1}), \beta^{-1}]\gamma f_\pi(\gamma^{-1}) \in D(\pi)T(e, f). \end{aligned}$$

Hence $D(\pi)T(e, \tilde{f}) \subset D(\pi)T(e, f)$.

Conversely, if $u \in D(\pi)T(e, f)$ then there exist $d \in D(\pi)$ and $\gamma \in \pi$ such that

$$\begin{aligned} u =& d\gamma f_\pi(\gamma^{-1}) = d\gamma\beta^{-1}\tilde{f}_\pi(\gamma^{-1})\beta \\ =& d[\gamma, \beta^{-1}][\beta^{-1}\gamma\tilde{f}_\pi(\gamma^{-1}), \beta]\gamma\tilde{f}_\pi(\gamma^{-1}) \in D(\pi)T(e, \tilde{f}). \end{aligned}$$

Hence

$$D(\pi)T(e, f) \subset D(\pi)T(e, \tilde{f}).$$

Thus we have

$$D(\pi)T(e, \tilde{f}) = D(\pi)T(e, f). \quad \square$$

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