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BIPARTITE POSETS WITH A UNIQUE OPTIMAL LINEAR EXTENSION

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1. Introduction

Let P be a finite poset and let $|P|$ be the *number of vertices* in P. A *subposet* of P is a subset of P with the induced order. A *chain* C in P is a subposet of P which is a linear order. The *length* of the chain C is $|C|-1$. A poset is *bipartite* if the length of each maximal chain is one. A *linear extension* of a poset P is a linear order $L = x_1, x_2, \ldots, x_n$ of the elements of P such that $x_i < x_j$ in P implies $i < j$.

Let P and Q be two disjoint posets. The *disjoint sum* $P + Q$ of P and Q is the poset on $P \cup Q$ such that $x < y$ if and only if $x, y \in P$ and $x < y$ in P or $x, y \in Q$ and $x < y$ in Q. The *linear sum* $P \oplus Q$ of P and Q is obtained from $P + Q$ by adding the relation $x < y$ for all $x \in P$ and $y \in Q$.

Throughout this section, L denotes an arbitrary linear extension of P. Let $a, b \in P$ with $a < b$. Then b *covers* a, denoted $a \prec b$, provided that for any $c \in P$, $a < c \leq b$ implies that $c = b$. A (P, L)*-chain* is a maximal sequence of elements $z_1, z_2, \ldots, z_k$ such that $z_1 \prec z_2 \prec \cdots \prec z_k$ in both L and P. Let $c(L)$ be the number of (P, L)-chains in L.

A consecutive pair (x_i, x_{i+1}) of elements in L is a *jump* (or *setup*) of P in L if x_i is not comparable to x_{i+1} in P. The jumps induce a decomposition $L = C_1 \oplus \cdots \oplus C_m$ of L into (P, L)-chains $C_1, \ldots, C_m$ where $m = c(L)$ and $(\max C_i, \min C_{i+1})$ is a jump of P in L for $i = 1, \ldots, m-1$. Let $s(L, P)$ be the number of jumps of P in L and let $s(P)$ be the minimum of $s(L, P)$ over all linear extensions L of P. The number $s(P)$ is called the *jump number* of P. If $s(L, P) = s(P)$, then

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L is called a *(jump) optimal linear extension* of P. We denote the set of all optimal linear extensions of P by $\mathcal{O}(P)$. If $|\mathcal{O}(P)| = |\mathcal{O}(Q)| = 1$, then $|\mathcal{O}(P \oplus Q)| = 1$ but $|\mathcal{O}(P + Q)| > 1$.

A *fence* on n elements is a bipartite poset $F_n = \{a_1 < a_2, a_2 > a_3, a_3 < a_4, \dots\}$. We know (See [2])

$$|\mathcal{O}(F_n)| = \begin{cases} 1, & \text{if } n \text{ is even} \\ 2^{(n-1)/2}, & \text{if } n \text{ is odd.} \end{cases}$$

2. Main Results

We denote the set of maximal [minimal] elements of a poset P by $Max(P)[Min(P)]$. Let $\mathcal{F}_{2n}$ be the family of bipartite posets P such that if $Max(P) = \{a_1, a_2, \dots, a_n\}$ and $Min(P) = \{b_1, b_2, \dots, b_n\}$, then $b_1 < a_1, a_1 > b_2, b_2 < a_2, \dots, b_n < a_n$ with possible comparabilities $a_i > b_j$ for some i, j where $1 \leq i < j - 1 \leq n - 1$.

THEOREM. *A bipartite poset P has a unique optimal linear extension if and only if $P \in \mathcal{F}_{2n}$ for some n.*

Proof. ($\Longrightarrow$) If P is not connected, then we have $|\mathcal{O}(P)| > 1$. Thus P is connected.

Case 1 : $|\{b \in Min(P) : a > b\}| \geq 2$ for every $a \in Max(P)$.

In this case, every optimal linear extension contains at least one singleton (P, L)-chain in $Min(P)$. If there are more than one singleton (P, L)-chains in $Min(P)$, then we have $|\mathcal{O}(P)| > 1$. If there is only one such singleton, then there exists $a \in Max(P)$ such that $|\{b \in Min(P) : a > b\}| = 2$, say $b_1 < a$, $b_2 < a$. Thus we have two optimal linear extensions $b_1 \oplus \{b_2, a\} \oplus \dots$ and $b_2 \oplus \{b_1, a\} \oplus \dots$, so $|\mathcal{O}(P)| > 1$.

Now, if Case 1 is not true, then one of the following two cases holds.

Case 2 : $|\{a \in Max(P) : |\{b \in Min(P) : a > b\}| = 1\}| \geq 2$.

In this case, we have $|\mathcal{O}(P)| > 1$.

Case 3 : $|\{a \in Max(P) : |\{b \in Min(P) : a > b\}| = 1\}| = 1$.

In this case, denote such elements by a_1, b_1. Now consider a poset $P - \{a_1, b_1\}$. If $P - \{a_1, b_1\}$ holds for Case 1 or Case 2 instead of P, then we have $|\mathcal{O}(P)| > 1$. If $P - \{a_1, b_1\}$ holds for Case 3, then denote such elements by a_2, b_2. Repeat the same process for $P - \{a_1, b_1, a_2, b_2\}$. By the same argument, we get P by $\{a_1, b_1, a_2, b_2, \dots, a_n, b_n\}$ for some n. Thus $P \in \mathcal{F}_{2n}$.

($\Longleftarrow$) For every $P \in \mathcal{F}_{2n}$, take a linear extension $L = \{b_n, a_n\} \oplus \{b_{n-1}, a_{n-1}\} \oplus \cdots \oplus \{b_1, a_1\}$. Then L is a unique optimal linear extension of $P \in \mathcal{F}_{2n}$.

By observing Theorem, we can count the number of $\mathcal{F}_{2n}$.

COROLLARY. *The number of $2n$ vertices bipartite posets with a unique optimal linear extension is $2^{\binom{n-1}{2}}$.*

Proof. By the construction of P in the proof of Theorem, we can add some comparabilities to $F_{2n} = \{b_1 < a_1, a_1 > b_2, b_2 < a_2, \dots, b_n < a_n\}$, say $a_i > b_j$ for some i, j such that $1 \le i < j - 1 \le n - 1$. Here, the number of possible choices for $\{i, j\}$ is $\binom{n-1}{2}$. By binomial theorem, we get the result.

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References

1. M. Chein and M. Habib, *The jump number of dags and posets: an introduction*, Ann. Disc. Math. **9** (1980), 189-194.
2. H. C. Jung, *Lower bounds of the number of jump optimal linear extensions: products of some posets*, Bull. Korean Math. Soc. **32** (1995), 171-177.

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