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MAÑÉ GENERICITY THEOREM FOR DIFFERENTIABLE MAPS

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1. Introduction

Smale [16] posed the following question; is having an attracting periodic orbit a generic property for diffeomorphisms of two-sphere S^2? (A *generic property* of $f \in Diff(M)$ is one that is true for a Baire set in $Diff(M)$). Mañé[5] and Plykin[13] had an positive answer for Axiom A diffeomorphisms of S^2. To explain our theorem, we begin by briefly recalling stability conjecture posed by palis and smale. Let M be a C^∞ closed manifold and let $C^1(M)$ be the space of C^1 maps endowed with C^1 topology. Then the set of all differomorphisms, $Diff^1(M)$, is open in $C^1(M)$. The following C^1 stability conjecture due to Palis and Smale [10], after long trial by mathematicians, was proved affirmatively on two dimensional manifolds by Mañé[6]. Aoki[1] and Hayashi[3] solved this problem for the general n-dimensional manifolds, independently. And Palis[8] proved the C^1-Ω-stability conjecture using techniques introduced by Mañé .

> 1.(C^1 stability conjecture)[4,10] If $f \in \mathcal{F}(M)$, then $\Omega(f)$ is a hyperbolic set, where $\mathcal{F}(M)$ denote the set of diffeomorphism $f \in Diff(M)$ that have a neighborhood $\mathcal{U}$ of f such that for all $g \in \mathcal{U}$ every periodic point of g is hyperbolic.

Morever, Moriyasu proved the same problem for C^1 maps on a C^∞ closed manifolds, as follows:

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11.(C^1 stability conjecture for C^1 maps)[7]. Suppose that $f \in C^1(M)$ satisfies the following condition: (S) $\Omega(f) \cap S(f) \subset \{p \in per(f) : p \text{ is sink}\}$. Then the following statements are equivalent;

(1) $f \in intP^1(M)$.
(2) f satisfies Axiom A and has no cycle.
(3) f is $C^1 - \Omega$ - inverse limit stability.

The above statement is very important for our theorem. From this theorem, our theorem includes the result of Mañé. To state our theorem, we begin by briefly recalling basic definitions and known facts. Let $P^1(M)$ be the set of $f \in C^1(M)$ such that every periodic point of f is hyperbolic. Let us define $P_i(f) = \{p \in per(f) : \dim E^s(p) = i\}$ for $f \in P^1(M)$ and $0 \leq i \leq \dim M$. Put

$$\mathcal{F}^1(M) = \{f \in intP^1(M) : f \text{ satisfies (S) in } C^1 \text{ stability conjecture for } C^1 \text{ maps}\}$$

When $f \in intP^1(M)$ is a diffeomorphism, it is known in Pliss[11] that the number of sink periodic points (i.e. $\#P_{\dim M}(f)$) and the number of source periodic points (i.e. $\#P_0(f)$) is finite, respectively. However, for the case of C^1 maps, it was proved by Moriyasu [7] that even though the number of source periodic points of f is not finite in general, the closure of set of source periodic points of f is hyperbolic.

When $\dim M = 2$, the goal of this paper is to prove the following generic properties of discrete dynamical systems using techniques closer to the Mañé original proof.

THEOREM. *Let M be a closed two dimensional manifold and let $C^1(M)$ be the space of C^1 maps on M endowed with C^1 topology. Then there exists a residual subset $\Theta \subset C^1(M)$ such that if $f \in \Theta$, then f has one of the following properties:*

(1) $f \in \mathcal{F}^1(M)$,
(2) *The number of sink periodic points of f is infinite,*
(3) *The closure of set of source periodic points of f is not hyperbolic, where $f \in \mathcal{F}^1(M)$ having the condition $\Omega(f) \cap S(f) \subset \{p \in per(f) : p \text{ is sink}\}$ satisfies Axiom A and has no cycles.*

The proof of our theorem is very similar to that of Mañé except for the definition of Axiom A for differentiable maps and its properties. Notice that all expanding maps belong to $intP^1(M)$ but they have infinitely many source periodic points.

First we establish some background. Let M be a C^∞ maninfold and let $\| \ \|$ be a Riemannian metric on the tangent TM. We denote as d the metric on M induced by $\| \ \|$. Also, the space of all C^1 maps from M into itself endowed with C^1topology denote $C^1(M)$. we denote as $S(f)$ *the set of singularities of* f i.e. those points of M where the derivative $D_xf : T_xM \to T_xM$ of $f : M \to M$ at x is not injective. Note that $S(f)$ is closed in M, and if $f \in Diff'(M)$ then $S(f) = \phi$ holds. If there is $n > 0$ such that $f^n(p) = p$ then $p \in M$ is called a *periodic point* of f. Denote as $\pi(f,p)$ the minimal integer n satisfying $f^n(p) = p$, and $\pi(f,p)$ is called a *period* of p. For $p \in per(f)$ with period $\pi(f,p)$, we say that p is called *hyperbolic* if the eigenvalus of $D_pf^{\pi(f,p)} : T_pM \to T_pM$ have moduli $\neq 1$. Suppose that $p \in per(f)$ is a hyperbolic point. Then there exist subspaces $E^s(p)$ and $E^u(p)$ of T_pM with $T_pM = E^s(p) \oplus E^u(p)$ such that

(a) $\quad D_pf^{\pi(f,p)}(E^s(p)) \subset E^s(p), \quad D_pf^{\pi(f,p)}(E^u(p)) = E^u(p)$

(b) $\quad$ there are $C > 0$ and $0 < \lambda < 1$ such that for every $n > 0$

$$\|Df^n \mid_{E^s} (p)\| \leq C\lambda^n, \ \ \|(Df^n \mid_{E^u} (p))^{-n}\| \leq C\lambda^n.$$

We say that p is a *saddle point* if $0 < \dim E^s(p) < \dim M$, a *sink point* if $\dim E^s(p) = \dim M$, a *source point* if $\dim E^s(p) = 0$. The key concept is that of hyperbolicity. The notions of a hyperbolic set generalize those mentioned above. We denote as $\mathbb{M}$ the product topological space $\Pi_{-\infty}^{\infty} M$. For $f \in C^1(M)$, we define a continuous map $\tilde{f} : \mathbb{M} \to \mathbb{M}$ by $\tilde{f}((x_i)) = (f(x_i))$ for $(x_i) \in \mathbb{M}$. For a natural projection $p^\circ : \mathbb{M} \to M$ defined by $p^\circ((x_i)) = x_0$, we have $p^\circ \circ \tilde{f} = f \circ p^\circ$. Put $\mathbb{M}_f = \{(x_i) \in \mathbb{M} : x_i \in M, f(x_i) = x_{i+1}, i \in \mathbb{Z}\}$ Then $\mathbb{M}_f$ is a closed $\tilde{f}$-invariant and $\tilde{f} \mid \mathbb{M}_f : \mathbb{M}_f \to \mathbb{M}_f$ is a homeomorphism. The dynamical system $(\mathbb{M}_f, \tilde{f})$ is called an *inverse limit system* of $(\mathbb{M}, f)$. We shall define the tangent space $T\mathbb{M}$ of $\mathbb{M}$. Put $T\mathbb{M} = \{(\tilde{x}, v) \in \mathbb{M} \times T\mathbb{M} : p^0(\tilde{x}) = \pi(v)\}$. Given $T\mathbb{M}$ the induced topology as a subset of $\mathbb{M} \times T\mathbb{M}$; then the natural projection $\tilde{\pi} : T\mathbb{M} \to \mathbb{M}, \pi(\tilde{x}, v) = \tilde{x}$ is a C^0 vector bundle. Norm on each fiber $\| \ \|$ is defined by $\|(\tilde{x}, v)\| = \|v\|$ for $(\tilde{x}. v) \in T\mathbb{M}$ and $\overline{p}^0 : T\mathbb{M} \to TM$ is defind $\overline{p}^0(\tilde{x}, v) = v$. It is known that $\overline{p}^0(T_{\tilde{x}}\mathbb{M}) = T_{x_0}\mathbb{M}$ and $\overline{p}^0 \mid T_{\tilde{x}}\mathbb{M} : T_{\tilde{x}}\mathbb{M} \to T_{x_0}\mathbb{M}$ is a linear isomorphism. For a subset Λ of M we can define $\Lambda_f = \{(x_i) \in \mathbb{M}_f : x_i \in \Lambda, i \in Z\}$.

Then Λ_f is an $\tilde{f}$-invariant. Put $T_{\tilde{x}}\mathbb{M} = \tilde{\pi}^{-1}(\tilde{x})$ for $\tilde{x} \in \mathbb{M}$ and define a subbundle $T\mathbb{M} |_{\Lambda_f}$ by $T\mathbb{M} |_{\Lambda_f} = \cup_{\tilde{x} \in \Lambda_f} T_{\tilde{x}}M$. In particular we put $T\mathbb{M}_f = T\mathbb{M} |_{\mathbb{M}_f}$. For $f \in C^1(M)$ define a linear bundle map

$$D\tilde{f} : T\mathbb{M} \to T\mathbb{M} \text{ by } D\tilde{f}(\tilde{x}, v) =(\tilde{f}(\tilde{x}), D_{x_0}f(v)) \\ =((x_i), v)) \text{ for } (\tilde{x}, v) \in T\mathbb{M}.$$

We say that a closed f-invarint set $\Lambda \subset M$ of $f \in C^1(M)$ is called a *hyperbolic set* (or we simply say: Λ is hyperbolic) if there exist a splitting $T\mathbb{M}_f \mid \Lambda_f = E^s \oplus E^u$ having the following properties:

(a) $D\tilde{f}(E^s) \subset E^s, \quad D\tilde{f}(E^u) = E^u$

(b) there exist $C > 0$ and $0 < \lambda < 1$ such that for every $n \geq 0$

$$\|D\tilde{f}^n |_{E^s} \| \leq C\lambda^n, \ \|(D\tilde{f}^n |_{E^u})^{-n}\| \leq C\lambda^n.$$

To define the concept of no cycle, we prepare the following sets: For $f \in C^1(M)$ and $\varepsilon > 0$, define a local stable set $W^s_\varepsilon(\tilde{x}, f)$ and a local unstable set $W^u_\varepsilon(\tilde{x}, f)$ of $\tilde{x} = (x_n) \in \mathbb{M}_f$ by:

$$W^s_\varepsilon(\tilde{x}, f) =\{y \in M : d(x_n, f^n(y)) \leq \varepsilon \text{ for every } n \geq 0\}$$
$$W^u_\varepsilon(\tilde{x}, f) =\{y \in M : \text{ there exists } \tilde{y} \in \mathbb{M}_f \text{ such that } y_0 = y \\ \text{and } d(x_{-n}, y_{-n}) \leq \varepsilon \text{ for every } n \geq 0\}$$

For $\tilde{x} = (x_n) \in \mathbb{M}_f$, define the following sets:

$$\textit{stable set } W^s(\tilde{x}, f) =\{y \in M : d(x_n, f^n(y)) \to 0 \text{ as } n \to \infty\}$$
$$\textit{unstable set } W^u(\tilde{x}, f) =\{y \in M : \text{there exists } \tilde{y} \in \mathbb{M}_f \text{ such that } y_0 = y \\ \text{and } d(x_{-n}, y_{-n}) \to 0 \text{ as } n \to \infty\}$$

Also we define the following sets for an f-invariant set $\Lambda \subset M$:

$$\begin{aligned} W^s(\Lambda, f) =& \{y \in M : d(\Lambda, f^n(y)) \to 0 \text{ as } n \to \infty\} \\ W^u(\Lambda, f) =& \{y \in M : \text{there exists } \tilde{y} \in \mathbb{M}_f \text{ such that } y_0 = y \\ & \text{and } d(\Lambda, y_{-n}) = 0 \text{ as } n \to \infty\} \end{aligned}$$

We say that an isolated hyperbolic set Λ of f is said to *have an n-cycle* if there exists a subset $\{\Lambda i_j : 1 \leq j \leq n+1\}$ of a finite decomposition $\{\Lambda_i : 1 \leq i \leq n+1\}$, each component is a basic set, such that the following are satisfied:

(1) $\Lambda_{i_1} = \Lambda_{i_{n+i}}$

(2) $\Lambda_{i_j} \neq \Lambda_{i_{j'}}$ $(1 \leq j \neq j' \leq n)$

(3) $\{W^s(\Lambda_{i_j}) - \Lambda_{i_j}\} \cap \{W^u(\Lambda_{i_{j+1}}) - \Lambda_{i_{j+1}}\} \neq \varnothing$ $(1 \leq j \leq n)$

For $f \in C^1(M)$ a *nonwandering set* $\Omega(f)$ is defined by

$$\begin{aligned} \Omega(f) =& \{x \in M : \text{for any neighborhood } U \text{ of } x \text{ there is } n > 0 \\ & \text{such that } f^n(U) \cap U \neq \varnothing\}. \end{aligned}$$

We say that f satisfies *Axiom A* if $\Omega(f)$ is a hyperbolic set and $\Omega(f) = \overline{Per(f)}$. For f satisfying Axiom A, if $\Omega(f)$ has no cycle then we say that *f has no cycle* .

For the proof of our theorem, we need the following Franks Lemma.

[Franks Lemma[3,7]] For any neighborhood $\mathcal{U}_0(f) \subset C^1(M)$ of $f \in C^1(M)$, there exists a neighborhood $\mathcal{U}_1(f) \subset \mathcal{U}_0(f)$ of f and $\varepsilon > 0$ such that for $g \in \mathcal{U}_1(f)$, a neighborhood $U \subset M$ of a finite sequence $\Phi = \{x_1, x_2, \ldots, x_N\}$ with $x_i \neq x_j$ $(i \neq j)$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ $(1 \leq i \leq N)$ with $\|L_i - D_{x_i}g\| \leq \varepsilon$ there are $\tilde{g} \in \mathcal{U}_1(f)$ and $\delta > 0$ satisfying the following properties:

(1) $B_\delta(\Phi) \subset U$

(2) $\tilde{g}(x) = g(x)$ for $x \in \Phi \cup \{M - B_{4\delta}(\Phi)\}$

(3) $D_{x_i}\tilde{g} = L_i$ for $i = 1, 2, \ldots, N$

$$B_\delta(\Phi) = \{y \in M : d(x_i, y) \leq \delta \quad \text{for some} \quad i\ \}$$

This was proved for the class of diffeomorphisms. But we remark that this lemma works for all differential maps.

2. Proof of Theorem

We denote as *Sink(f)* the set of sink periodic points of a differentiable map f and as *Source(f)* the set of sink periodic points respectively. For $f \in C^1(M)$ let $\overline{\Lambda}(f)$ be the closure of the union of *Sink(f)* and *Source(f)*. Then the set valued function $\overline{\Lambda} : C^1(M) \to 2^M, f \to \overline{\Lambda}(f)$ is lower semicontinuous. To get this, let $\{g_1, g_2, ...\}$ be the sequence of $C^1(M)$ with $g_n \to f$, $(n \to \infty)$, and $x_0 \in \overline{\Lambda(f)}$ and we show that there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $q_k \in \overline{\Lambda}(g_{n_k})$ such that $q_k \to x_0, (k \to \infty)$. As $x_0 \in \overline{\Lambda}(f)$ there are sink or source periodic points $p_1, p_2, ...$ of f such that$p_k \in B_{1/k}(x_0)$ for every $k = 1, 2, ...,$ where $B_r(x)$ denotes the closed r -ball centered at $x \in M$. For $k = 1, 2, \ldots$, take small neighborhoods U_k of p_k and $\mathcal{U}_k$ of f, and define $F_k : \mathcal{U}_k \times U_k \to T_{p_k}M$ by $F_k(g, y) = \exp_{p_k}^{-1} o g^{\pi(p_k, f)}(y) - \exp_{p_k}^{-1}$, where $\exp_x$ denotes the exponential map at $x \in M$. Then by applying the implicit function theorem for F_k, We get $q_k \in \overline{\Lambda}(g_{n_k}) \cap B_{\frac{1}{k}}(p_k) \subset \overline{\Lambda}(g_{n_k}) \cap B_{\frac{2}{k}}(x_0)$ for sufficiently large $n_k \in \mathbb{N}$. Thus $q_k \to x_0, (k \to \infty)$ and so $\overline{\Lambda}$ is lower semicontinuous.

Therefore, there is a residual set $\Theta \subset C^1(M)$ such that $\overline{\Lambda} : C^1(M) \to 2^M$ is continuous at every $f \in \Theta$.

We claim that every $f \in \Theta$ satisfies one of the three properties of our theorem. Suppose that $f \in \Theta$ dose not satisfy (2) and (3). Let $p_1, p_2, \ldots, p_N$ be all elements of $Sink(f)$ and $\varepsilon_1 > 0$ be a constant which satisfying $B_{\varepsilon_1}(p_i) \cap B_{\varepsilon_1}(p_j) = \varnothing$ for each $1 \leq i < j \leq N$. As $\overline{\Lambda} = \{p_1, p_2, \ldots, p_N\} \cup \overline{Source(f)}$ is hyperbolic, we can take a Riemannian metric $\|\ \|$ and $0 < \lambda < 1$ such that

$$\|D_x f(v)\| > \lambda^{-1}\|v\| \quad \text{for every} \quad 0 \neq v \in T_x M, \quad x \in \overline{Source(f)},$$

$$\|D_{p_i} f^{\pi(p_i,f)}(u)\| < \lambda^{\pi(p_i,f)}\|u\| \quad \text{for every} \quad 0 \neq u \in T_{p_i}M, \quad i = 1, 2, \ldots, N.$$

Let $\lambda < \lambda_1 < 1$. Since $f \in \Theta$, we can choose a small neighborhood $\mathcal{U}_1(f)$ of f as follows. For any $g \in \mathcal{U}_1(f)$,

$$\|D_y g(v)\| > \lambda_1^{-1}\|v\| \quad \text{for every} \quad 0 \neq v \in T_yM, \quad y \in \overline{Source(g)},$$

and there exist $r_1, r_2, \ldots, r_N \in M$ with $r_i \in B_{\varepsilon_1}(p_i)(i = 1, 2, \ldots, N)$ such that $Sink(g) = \{r_1, r_2, \ldots, r_N\}$, $\pi(r_i, g) = \pi(p_i, f)$ and

$$\|D_{r_i} g^{\pi(r_i,g)}(u)\| < \lambda_1^{\pi(r_i,g)}\|u\| \text{ for every } 0 \neq u \in T_{r_i}M, \ i = 1, 2, \ldots, N.$$

If f dose not satisfy (1) then one of the following porperties is satisfied.

(a) There are $g_1, g_2, \ldots \in C^1(M)$ and $q_n \in Per(g_n)$, $(n = 1, 2, ..)$ such that $g_n \to f \quad (n \to \infty)$ and maximal moduli of eigenvalues of $D_{q_n} g_n^{\pi(q_n,g_n)}$ equals to 1.

(b) There are $g_1, g_2, \ldots \in C^1(M)$ and $q_n \in Per(g_n)$, $(n = 1, 2, ..)$ such that $g_n \to f \quad (n \to \infty)$ and minimal moduli of eigenvalues of $D_{q_n} g_n^{\pi(q_n,g_n)}$ equals to 1.

If we have (a), then by using Franks lemma we construct $\tilde{g}_n \in \mathcal{U}_1(f)$ such that $q_n \in Source(\tilde{g}_n)$, $\pi(q_n, \tilde{g}_n) = \pi(q_n, g_n)$, $\|v_n\| < \|D_{q_n} \tilde{g}_n^{\pi(q_n,\tilde{g}_n)} (v_n)\| < \lambda_1^{-\pi(q_n,\tilde{g}_n)}\|v_n\|$ for some $0 \neq v_n \in T_{q_n}M$ and large n. On the other hand, $q_n, \tilde{g}_n(q_n), \ldots, \tilde{g}_n^{\pi(q_n,\tilde{g}_n)-1}(q_n) \in Source(\tilde{g}_n)$. This is a contradiction.

If we have (b), then we can construct $\tilde{g}_n \in \mathcal{U}_1(f)$ such that $q_n \in Sink(\tilde{g}_n)$, $\pi(q_n, \tilde{g}_n) = \pi(q_n, g_n)$, $\lambda_1^{\pi(q_n,\tilde{g}_n)}\|v_n\| < \|D_{q_n}\tilde{g}_n^{\pi(q_n,\tilde{g}_n)}(v_n)\| < \|v_n\|$ for some $0 \neq v_n \in T_{q_n}M$ and large n. This contradicts the definition of λ_1 and $\mathcal{U}_1(f)$, finishing the proof.

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