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BEST SIMULTANEOUS APPROXIMATIONS IN A NORMED LINEAR SPACE

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1. Introduction

We characterize best simultaneous approximations from a finite-dimensional subspace of a normed linear space. In the characterization we reveal usefulness of a minimax theorem presented in [2,4].

We present this minimax theorem and some corollaries in [4]. In [3], [4], [5] and [6], we can find characterizations of best uniform approximations and best simultaneous approximations from a finite-dimensional subspace of continuous functions from a compact Hausdorff space to a normed linear space. Next, we give a characterization of best simultaneous approximations from a finite-dimensional subspace of a normed linear space. Finally, we give a characterization of best simultaneous approximations from a convex set in a finite-dimensional subspace of a normed linear space.

Let U and V be nonempty compact convex subsets of two Hausdorff topological vector spaces. Suppose that a function $J : U \times V \to \mathbb{R}$ is such that for each $v \in V$, $J(\cdot, v)$ is lower semi-continuous and convex on U, and for each $u \in U$, $J(u, \cdot)$ is upper semi-continuous and concave on V. Then, as is well known [2], there exists a saddle point $(u^*, v^*) \in U \times V$ such that

$$J(u^*, v) \le J(u^*, v^*) \le J(u, v^*), \quad u \in U,\ v \in V, \tag{1-1}$$

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that is,

$$\min_{u\in U}\max_{v\in V} J(u,v) = \max_{v\in V}\min_{u\in U} J(u,v).$$

However, if the set V is not convex, or if for some $u \in U$, $J(u,\cdot)$ is not a concave function on V, the relation (1-1) does not hold in general.

We present here a generalized minimax theorem that holds even under these conditions.

Let U be a nonempty compact convex subset of a Hausdorff topological vector space, and let V be an arbitrary nonempty set. Suppose that $J : U \times V \to \mathbb{R}$ is such that for each $v \in V$, $J(\cdot, v)$ is a lower semi-continuous and convex function on U. For each positive integer p, define the set

$$\overline{V}_p = \{(\overline{\lambda}_p, \overline{v}_p) | \overline{\lambda}_p = (\lambda_1, \cdots, \lambda_p), \overline{v}_p = (v_1, \cdots, v_p),$$

$$\sum_{i=1}^{p} \lambda_i = 1, \lambda_i \geq 0, v_i \in V (i = 1, \cdots, p)\}.$$

THEOREM 1.1. [4]. *Let U be an n-dimensional, compact convex subset of a Hausdorff topological vector space ($n \geq 1$), and let V be a compact Hausdorff space. Let $J : U \times V \to \mathbb{R}$ be a jointly continuous function. Then $u^* \in U$ minimizes $\max_{v\in V} J(u,v)$ over U if and only if there exists $(\overline{\lambda}^*_{n+1}, \overline{v}^*_{n+1}) \in \overline{V}_{n+1}$ such that*

$$\sum_{i=1}^{n+1} \lambda_i J(u^*, v_i) \leq \sum_{i=1}^{n+1} \lambda_i^* J(u^*, v_i^*) \leq \sum_{i=1}^{n+1} \lambda_i^* J(u, v_i^*)$$

holds for all $(\overline{\lambda}_{n+1}, \overline{v}_{n+1}) \in \overline{V}_{n+1}$ and for all $u \in U$.

2. The best simultaneous approximation in a normed linear space

Let X be a normed linear space and let K be an n-dimensional subspace of X. Suppose that $x_1, \cdots, x_\ell$ are in X. The problem is to find an element $k_o \in K$ which minimizes

$$\max_{1\leq j\leq \ell} ||x_j - k|| \tag{2-1}$$

over the subspace K. If such an element k_o in K exists, we call it a best simultaneous approximation for $(x_1, \cdots, x_\ell)$ from K.

Remark that

$$\max_{1 \le j \le \ell} ||x_j - k|| = \max_{\mathbf{a} \in A} || \sum_{j=1}^{\ell} a_j x_j - k||,$$

where the set A is defined by

$$A = \{\mathbf{a} = (a_1, \cdots, a_\ell) | \sum_{j=1}^{\ell} a_j = 1, a_j \ge 0 (1 \le j \le \ell)\}.$$

This follows from the expression

$$\sum_{j=1}^{\ell} a_j x_j - k = \sum_{j=1}^{\ell} a_j (x_j - k)$$

and the inequalities

$$\begin{aligned} \max_{1 \le j \le \ell} ||y_j|| &\le \max_{\mathbf{a} \in A} || \sum_{j=1}^{\ell} a_j y_j|| \\ &\le \max_{\mathbf{a} \in A} \sum_{j=1}^{\ell} a_j ||y_j|| \\ &\le \max_{1 \le j \le \ell} ||y_j||. \end{aligned}$$

Then (2-1) can be expressed as

$$\max_{\mathbf{a} \in A} || \sum_{j=1}^{\ell} a_j x_j - k||.$$

Thus the problem takes on the expression

$$\text{minimize } \max_{\mathbf{a} \in A} || \sum_{j=1}^{\ell} a_j x_j - k|| \text{ over the set } K.$$

Note that the set A is compact.

THEOREM 2.1. *Let K be an n-dimensional subspace of a normed linear space X and $x_1, \cdots, x_\ell \in X$. Then $k_o \in K$ is a best simultaneous approximation for $\mathbf{X} = (x_1, \cdots, x_\ell)$ from K if and only if there exist $\lambda_1^*, \cdots, \lambda_p^* > 0$, $\sum_{i=1}^p \lambda_i^* = 1$, and p vectors $\mathbf{a}_1^*, \cdots, \mathbf{a}_p^* \in A$, where $1 \leq p \leq n+1$, such that*

$$\text{(i)} \qquad ||\sum_{j=1}^{\ell} a_{ij}^* x_j - k_o|| = \max_{1 \leq j \leq \ell} ||x_j - k_o|| \quad i = 1, \cdots, p;$$

$$\sum_{i=1}^{p} \lambda_i^* ||\sum_{j=1}^{\ell} a_{ij}^* x_j - k|| \geq \max_{1 \leq j \leq \ell} ||x_j - k_o||$$

$$\text{(ii)} \qquad = \sum_{i=1}^{p} \lambda_i^* ||\sum_{j=1}^{\ell} a_{ij}^* x_j - k_o||$$

for any $k \in K$.

Proof. ($\Rightarrow$) Let $k_o \in K$ be a best simultaneous approximation for $\mathbf{X} = (x_1, \cdots, x_\ell)$ from K and let

$$U = \{k \in K \mid ||k_o - k|| \leq 1\}.$$

Note that U is a compact subset of K. Applying Theorem 1.1 yields the existence of $\lambda_1', \cdots, \lambda_{n+1}' \geq 0$, $\sum_{i=1}^{n+1} \lambda_i' = 1$, and $\mathbf{a}_1^*, \cdots, \mathbf{a}_{n+1}^* \in A$ such that

$$\text{(2-2)} \qquad \sum_{i=1}^{n+1} \lambda_i' ||k_o - \sum_{j=1}^{\ell} a_{ij}^* x_j|| \geq \sum_{i=1}^{n+1} \lambda_i ||k_o - \sum_{j=1}^{\ell} a_{ij} x_j||$$

for any $(\lambda_1, \cdots, \lambda_{n+1})$ with $\sum_{i=1}^{n+1} \lambda_i = 1$ and $\lambda_i \geq 0$, $i = 1, \cdots, n+1$, and $a_{ij} \geq 0$, $j = 1, \cdots, \ell$, $\sum_{j=1}^{\ell} a_{ij} = 1$;

$$\text{(2-3)} \qquad \sum_{i=1}^{n+1} \lambda_i' ||k - \sum_{j=1}^{\ell} a_{ij}^* x_j|| \geq \sum_{i=1}^{n+1} \lambda_i' ||k_o - \sum_{j=1}^{\ell} a_{ij}^* x_j||$$

for $k \in U$.

Let us denote by $\lambda_1^*, \cdots, \lambda_p^*$ the nonzero elements within $\lambda_1', \cdots, \lambda_{n+1}'$ and by $\mathbf{a}_1^*, \cdots, \mathbf{a}_p^*$ the corresponding elements within $\mathbf{a}_1^*, \cdots, \mathbf{a}_{n+1}^*$. The assertion (i) follows from (2-2) which means, for $i = 1, \cdots, p$,

$$||\sum_{j=1}^{\ell} a_{ij}^* x_j - k_o|| = \max_{1\le j\le \ell} ||x_j - k_o||.$$

On the other hand, it follows from (2-3) that

$$\sum_{i=1}^{p} \lambda_i^* ||k - \sum_{j=1}^{\ell} a_{ij}^* x_j|| \ge \sum_{i=1}^{p} \lambda_i^* ||k_o - \sum_{j=1}^{\ell} a_{ij}^* x_j||$$

holds for any $k \in U$. Since the left-hand side is a convex function of k and has a local minimum at k_o, k_o realizes a global minimum by a property of convex functions. Thus (ii) follows.

($\Leftarrow$) Conversely, suppose that (i) and (ii) hold. Let $\overline{V}_p = \{\overline{\lambda}_p | \overline{\lambda}_p = (\lambda_1, \cdots, \lambda_p), \sum_{i=1}^{p} \lambda_i = 1, \lambda_i \ge 0 \, (i = 1, \cdots, p)\}$. These two conditions yield

$$\begin{aligned}\sup_{(\overline{\lambda}_p, \overline{\mathbf{a}}) \in \overline{V}_p \times A} \inf_{k\in K} \sum_{i=1}^{p} \lambda_i ||k - \sum_{j=1}^{\ell} a_{ij} x_j|| &\ge \sum_{i=1}^{p} \lambda_i^* ||k_o - \sum_{j=1}^{\ell} a_{ij}^* x_j|| \\ &= \max_{1\le j\le \ell} ||x_j - k_o||\end{aligned}$$

and

$$\begin{aligned}&\sup_{(\overline{\lambda}_p, \overline{\mathbf{a}}) \in \overline{V}_p \times A} \inf_{k\in K} \sum_{i=1}^{p} \lambda_i ||k - \sum_{j=1}^{\ell} a_{ij} x_j|| \\ &\qquad \le \inf_{k\in K} \max_{(\overline{\lambda}_p, \overline{\mathbf{a}}_p) \in \overline{V}_p \times A} \sum_{i=1}^{p} \lambda_i ||k - \sum_{j=1}^{\ell} a_{ij} x_j|| \\ &\qquad = \inf_{k\in K} \max_{1\le j\le \ell} ||x_j - k||.\end{aligned}$$

Therefore k_o is a best simultaneous approximation for $\mathbf{X} = (x_1, \cdots, x_\ell)$ from K.

Let S be a compact Hausdorff space and T a normed linear space with $||\cdot||$. Let $C(S,T)$ denote the set of all continuous functions from S to T and let K be an n-dimensional subspace of $C(S,T)$. For $f \in C(S,T)$, we define the uniform norm of f by

$$|||f||| = \max_{x\in S} ||f(x)||$$

and endow the linear space $C(S,T)$ with the uniform topology. Suppose that $f_1, \cdots, f_\ell$ are in $C(S,T)$.

Furthermore, if we regard the set A as the set of ℓ-dimensional row vectors and denote by $\mathbf{F}(x)$ the column vector $(f_1(x), f_2(x), \cdots, f_\ell(x))^t$, $\sum_{j=1}^{\ell} a_j f_j(x)$ can be denoted by the inner product $\mathbf{a}\mathbf{F}(x)$ of two vectors $\mathbf{a}$ and $\mathbf{F}(x)$. By using Theorem 2.1, we get results for a continuous function space. Next we give a theorem in [5].

THEOREM 2.2 [5]. *A function $f^* \in K$ is a best simultaneous approximation for $(f_1, \cdots, f_\ell)$. if and only if there exist $\lambda_1^*, \cdots, \lambda_k^* > 0, \sum_{i=1}^{k} \lambda_i^* = 1$, k distinct elements $s_1^*, \cdots, s_k^* \in S$, and k vectors $\mathbf{a}_1^*, \cdots, \mathbf{a}_k^* \in A$, where $1 \le k \le n+1$, such that*

(i)
$$\begin{aligned} ||\mathbf{a}_i^*\mathbf{F}(s_i^*) - f^*(s_i^*)|| &= \max_{1\le j\le \ell} ||f_j(s_i^*) - f^*(s_i^*)|| \\ &= \max_{1\le j\le \ell} |||f_j - f^*||| \quad i = 1, \cdots, k; \end{aligned}$$

(ii)
$$\sum_{i=1}^{k} \lambda_i^* ||\mathbf{a}_i^*\mathbf{F}(s_i^*) - f(s_i^*)|| \ge \sum_{i=1}^{k} \lambda_i^* ||\mathbf{a}_i^*\mathbf{F}(s_i^*) - f^*(s_i^*)||$$

for all $f \in K$.

COROLLARY 2.3. *Let K be an n-dimensional subspace of $C(S,T)$ and $f_1, f_2, \cdots, f_\ell \in C(S,T)$. Then the following are equivalent:*

(1) $f^* \in K$ is a best simultaneous approximation for $\mathbf{F} = (f_1, \cdots, f_\ell)$.

(2) There exist $\lambda_1^*, \cdots, \lambda_p^* > 0$, $\sum_{i=1}^{p} \lambda_i^* = 1$, and p vectors $\mathbf{a}_1^*, \cdots, \mathbf{a}_p^*$

$\in A$, *where* $1 \le p \le n+1$, *such that*

(i) $$|||\sum_{j=1}^{\ell} a_{ij}^* f_j - f^*||| = \max_{1\le j\le \ell} |||f_j - f^*|||, \ i = 1, \cdots, p;$$

(ii) $$\sum_{i=1}^{p} \lambda_i^* |||\sum_{j=1}^{\ell} a_{ij}^* f_j - f||| \ge \sum_{i=1}^{p} \lambda_i^* |||\sum_{j=1}^{\ell} a_{ij}^* f_j - f^*|||$$

or

$$\sum_{i=1}^{k} \lambda_i^* |||\mathbf{a}_i^* \mathbf{F} - f||| \ge \sum_{i=1}^{k} \lambda_i^* |||\mathbf{a}_i^* \mathbf{F} - f^*|||$$

for any $f \in K$.

(3) *There exist* $\lambda_1^*, \cdots, \lambda_k^* > 0, \sum_{i=1}^{k} \lambda_i^* = 1$, k *distinct elements* $s_1^*, \cdots, s_k^* \in S$, *and* k *vectors* $\mathbf{a}_1^*, \cdots, \mathbf{a}_k^* \in A$, *where* $1 \le k \le n+1$, *such that*

(i) $$||\mathbf{a}_i^* \mathbf{F}(s_i^*) - f^*(s_i^*)|| = \max_{1\le j\le \ell} ||f_j(s_i^*) - f^*(s_i^*)|| = \max_{1\le j\le \ell} |||f_j - f^*|||, \ i = 1, \cdots, k;$$

(ii)

$$\sum_{i=1}^{k} \lambda_i^* ||\mathbf{a}_i^* \mathbf{F}(s_i^*) - f(s_i^*)|| \ge \sum_{i=1}^{k} \lambda_i^* ||\mathbf{a}_i^* \mathbf{F}(s_i^*) - f^*(s_i^*)||$$

for all $f \in K$.

Proof. By Theorem 2.1, (1) and (2) are equivalent. By Theorem 2.2, (1) and (3) are equivalent.

3. Best simultaneous approximation from a convex set

Let X be a normed linear space and let C be a convex subset of a finite-dimensional subspace of X. Suppose that $x_1, \cdots, x_\ell$ are in X. The problem is to find an element $c_o \in C$ which minimizes

(3-1) $$\max_{1\le j\le \ell} ||x_j - k||$$

over the convex set C. If such an element c_o in C exists, we call it a best simultaneous approximation for $\mathbf{X} = (x_1, \cdots, x_\ell)$ from C. As in the section 2, we get

$$\max_{1 \le j \le \ell} ||x_j - k|| = \max_{\mathbf{a} \in A} ||\sum_{j=1}^{\ell} a_j x_j - k||,$$

where the set A is the same set as in the section 2. Then the above problem takes on the expression

$$\text{minimize } \max_{\mathbf{a} \in A} ||\sum_{j=1}^{\ell} a_j x_j - k|| \text{ over the set } C.$$

By the same argument as in the proof of Theorem 2.1, we can prove the following Theorem 3.1.

THEOREM 3.1. *Let C be a convex subset in an n-dimensional subspace of a normed linear space X and $x_1, \cdots, x_\ell \in X$. Then $c_o \in C$ is a best simultaneous approximation for $\mathbf{X} = (x_1, \cdots, x_\ell)$ from C if and only if there exist $\lambda_1^*, \cdots, \lambda_p^* > 0$, $\sum_{i=1}^{p} \lambda_i^* = 1$, and p vectors $\mathbf{a}_1^*, \cdots, \mathbf{a}_p^* \in A$, where $1 \le p \le n+1$, such that*

(i) $$||\sum_{j=1}^{\ell} a_{ij}^* x_j - c_o|| = \max_{1 \le j \le \ell} ||x_j - c_o|| \quad i = 1, \cdots, p;$$

$$\sum_{i=1}^{p} \lambda_i^* ||\sum_{j=1}^{\ell} a_{ij}^* x_j - c|| \ge \max_{1 \le j \le \ell} ||x_j - c_o||$$

(ii) $$= \sum_{i=1}^{p} \lambda_i^* ||\sum_{j=1}^{\ell} a_{ij}^* x_j - c_o||$$

for any $c \in C$.

By using Theorem 3.1, we get a result for a continuous function space. Next we give a theorem in [3].

THEOREM 3.2 [3]. *Let C be a convex subset in an n-dimensional subspace of $C(S,T)$ and $f_1, \cdots, f_\ell \in C(S,T)$. Then a function $f^* \in C$ is a best simultaneous approximation for $\mathbf{F} = (f_1, \cdots, f_\ell)$ if and only if there exist $\lambda_1^*, \cdots, \lambda_k^* > 0, \sum_{i=1}^k \lambda_i^* = 1$, k distinct elements $s_1^*, \cdots, s_k^* \in S$, and k vectors $\mathbf{a}_1^*, \cdots, \mathbf{a}_k^* \in A$, where $1 \le k \le n+1$, such that*

$$\text{(i)} \qquad \|\mathbf{a}_i^*\mathbf{F}(s_i^*) - f^*(s_i^*)\| = \max_{1\le j\le \ell} \|f_j(s_i^*) - f^*(s_i^*)\| = \max_{1\le j\le \ell} |||f_j - f^*|||, \; i = 1, \cdots, k;$$

(ii)

$$\sum_{i=1}^k \lambda_i^* \|\mathbf{a}_i^*\mathbf{F}(s_i^*) - f(s_i^*)\| \ge \sum_{i=1}^k \lambda_i^* \|\mathbf{a}_i^*\mathbf{F}(s_i^*) - f^*(s_i^*)\|$$

for all $f \in C$.

The next corollary states that f^* is a best simultaneous approximation on S if and only if it also is on some finite set of S.

COROLLARY 3.3. *Let C be a convex subset in an n-dimensional subspace of $C(S,T)$. Then the following are equivalent:*

(1) $f^* \in C$ is a best simultaneous approximation for $(f_1, \cdots, f_\ell)$.

(2) There exist $\lambda_1^*, \cdots, \lambda_p^* > 0, \sum_{i=1}^p \lambda_i^* = 1$, and p vectors $\mathbf{a}_1^*, \cdots, \mathbf{a}_p^* \in A$, where $1 \le p \le n+1$, such that

$$\text{(i)} \qquad |||\sum_{j=1}^{\ell} a_{ij}^* f_j - f^*||| = \max_{1\le j\le \ell} |||f_j - f^*|||, \; i = 1, \cdots, p;$$

$$\text{(ii)} \qquad \sum_{i=1}^p \lambda_i^* |||\sum_{j=1}^{\ell} a_{ij}^* f_j - f||| \ge \sum_{i=1}^p \lambda_i^* |||\sum_{j=1}^{\ell} a_{ij}^* f_j - f^*|||$$

for any $f \in C$.

(3) There exist $\lambda_1^*, \cdots, \lambda_k^* > 0, \sum_{i=1}^k \lambda_i^* = 1$, k distinct elements $s_1^*, \cdots, s_k^* \in S$, and k vectors $\mathbf{a}_1^*, \cdots, \mathbf{a}_k^* \in A$, where $1 \le k \le n+1$,

such that

(i)
$$\|\mathbf{a}_i^*\mathbf{F}(s_i^*) - f^*(s_i^*)\| = \max_{1\le j\le \ell} \|f_j(s_i^*) - f^*(s_i^*)\|$$
$$= \max_{1\le j\le \ell} |||f_j - f^*|||, \ i = 1, \cdots, k;$$

(ii)
$$\sum_{i=1}^{k} \lambda_i^* \|\mathbf{a}_i^*\mathbf{F}(s_i^*) - f(s_i^*)\| \ge \sum_{i=1}^{k} \lambda_i^* \|a_i^*\mathbf{F}(s_i^*) - f^*(s_i^*)\|$$
$$= \sum_{i=1}^{k} \lambda_i^* \|a_i^*\mathbf{F}(s_i^*) - f^*(s_i^*)\|$$

for all $f \in C$.

Proof. By Theorem 3.1, (1) and (2) are equivalent. By Theorem 3.2, (1) and (3) are equivalent.

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