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ALMOST DERIVATIONS ON THE BANACH ALGEBRA $C^n[0,1]$

KIL-WOUNG JUN AND DAL-WON PARK*

1. Introduction

A linear map T from a Banach algebra A into a Banach algebra B is *almost multiplicative* if $\|T(fg) - T(f)T(g)\| \leq \epsilon\|f\|\|g\|\,(f, g \in A)$ for some small positive ϵ. B. E. Johnson [**4**, **5**] studied whether this implies that T is near a multiplicative map in the norm of operators from A into B. K. Jarosz [**2**, **3**] raised the conjecture : If T is an almost multiplicative functional on uniform algebra A, there is a linear and multiplicative functional F on A such that $\|T - F\| \leq \epsilon'$, where $\epsilon' \to 0$ as $\epsilon \to 0$. B. E. Johnson [4] gave an example of non-uniform commutative Banach algebra which does not have the property described in the above conjecture. He proved also that $C(K)$ algebras and the disc algebra $A(D)$ have this property [5]. We extend this property to a derivation on a Banach algebra.

Let $\mathcal{A}$ be a commutative Banach algebra with unit. A Banach $\mathcal{A}$-module is a Banach space $\mathcal{M}$ together with a continuous homomorphism $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{M})$. A *derivation*, or a *module derivation*, of $\mathcal{A}$ into $\mathcal{M}$ is a linear map $D : \mathcal{A} \to \mathcal{M}$ which satisfies the identity

$$D(fg) = \rho(f)D(g) + \rho(g)D(f), \quad f, g \in \mathcal{A}.$$

In this paper we show that there exists a continuous derivation near a continuous almost derivation on a Banach algebra of differentiable functions.

We now give a precise definition of almost derivation.

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DEFINITION 1. A linear map $D:\mathcal{A}\to\mathcal{M}$ is an ϵ-almost derivation, or an ϵ-almost module derivation if D satisfies

$$\|D(fg)-\rho(f)D(g)-\rho(g)D(f)\|\leq\epsilon\|f\|\|g\|,\quad f,g\in\mathcal{A}.$$

DEFINITION 2. A linear map $D:\mathcal{A}\to\mathcal{M}$ is a strong ϵ-almost derivation, or a strong ϵ-almost module derivation if D satisfies

$$\|D(fg)-\rho(f)D(g)-\rho(g)D(f)\|\leq\epsilon\|fg\|,\quad f,g\in\mathcal{A}.$$

Note that if $D:\mathcal{A}\to\mathcal{M}$ is a strong ϵ-almost derivation, then D is an ϵ-almost derivation. Let D be a derivation on a Banach algebra $\mathcal{A}$. If F is a linear map on $\mathcal{A}$ such that

$$\|D(f)-F(f)\|\leq\epsilon\|f\|,\quad f\in\mathcal{A},$$

then it is easy to show that F is an ϵ-almost derivation on $\mathcal{A}$.

Let $C^n[0,1]$ denote the algebra of all complex-valued functions on $[0,1]$ which have n continuous derivatives. It is well known that $C^n[0,1]$ is a Banach algebra under the norm

$$\|f\|_n=\max_{t\in[0,1]}\sum_{k=0}^{n}|f^{(k)}(t)|/k!.$$

Assume that $\mathcal{M}$ is a Banach $C^n[0,1]$-module. We set $z(t)=t,\ 0\leq t\leq 1$. The *differential subspace* is the set $\mathcal{W}$ of all vectors m in $\mathcal{M}$ such that the map $p\to\rho(p')m$ is continuous on $\mathcal{P}$, where $\mathcal{P}$ is the dense subalgebra of polynomials in z. It is clear that $\mathcal{W}$ is a linear subspace of $\mathcal{M}$ and $m\in\mathcal{W}$ iff $|||m|||=\sup\{\|\rho(p)m\|:\|p\|_{n-1}=1\}<\infty$.

EXAMPLE. Let $\rho:C^1[0,1]\to\mathcal{B}(\mathcal{C})$ be defined by $\rho(f)=f(0)$ where $\mathcal{C}$ is the complex number field. Then $\mathcal{C}$ is a Banach $C^1[0,1]$-module. We define $D:C^1[0,1]\to\mathcal{C}$ by $D(f)=f'(0)+f(0)\epsilon$. It is easy to see that D is a strong ϵ-almost derivation on $C^1[0,1]$. We put $F(f)=f'(0),\quad f\in C^1[0,1]$. Then F is a derivation such that $|D(f)-F(f)|\leq\epsilon|f|,\quad f\in C^1[0,1]$.

We need the following result from [1] to prove our main theorem.

THEOREM 3. *Let $\mathcal{M}$ be a $C^n[0,1]$-module with differential subspace $\mathcal{W}$. Then*

(1) $\|m\| \leq |||m|||, \quad m \in \mathcal{W}$.
(2) *$\mathcal{W}$ is a Banach space with respect to the norm $|||\cdot|||$.*
(3) *$\mathcal{W}$ is a $C^{n-1}[0,1]$- module. There exists a unique continuous homomorphism $\gamma : C^{n-1}[0,1] \to \mathcal{B}(\mathcal{W})$ such that*

$$\gamma(p)m = \rho(p)m, \quad m \in \mathcal{W}, \quad p \in \mathcal{P}.$$

2. Results

In this section we denote $\|f\|_n$ by $\|f\|$, $\quad f \in C^n[0,1]$. Recall that the *ascent* of eigenvalue λ for a linear operator T is the smallest integer k such that $(T-\lambda I)^{k+1}x = 0$ implies $(T-\lambda I)^k x = 0$. We first consider that a strong ϵ-almost derivation D from $C^n[0,1]$ into a $C^n[0,1]$-module $\mathcal{M}$ is near a derivation.

THEOREM 4. *Let $\mathcal{M}$ be a finite dimensional Banach $C^n[0,1]$-module. If $D : C^n[0,1] \to \mathcal{M}$ is a continuous strong ϵ-almost derivation and the ascent of every eigenvalue for $\rho(z)$ less than $n/2$ then there exists a continuous derivation $F : C^n[0,1] \to \mathcal{M}$ such that*

$$\|D(f) - F(f)\| \leq \epsilon' \|f\|, \quad f \in C^n[0,1]$$

where $\epsilon' \to 0$ as $\epsilon \to 0$.

Proof. By description of [1] for the derivations from $C^n[0,1]$ to a finite dimensional Banach $C^n[0,1]$-module $\mathcal{M}$, we can suppose that $\rho(z)$ has a single eigenvalue λ_0 on $\mathcal{M}$ and that $\lambda_0 = 0$ for simplicity. A further simplification is possible, and so we suppose $\mathcal{M} = sp\{m_0, \rho(z)m_0, ..., \rho(z)^k m_0\}$ where m_0 is a fixed vector and $2k+2 \leq n$. With respect to this basis, the operator $\rho(f)(f \in C^n[0,1])$ has the matrix

$$\begin{pmatrix} \delta_0(f) & 0 & 0 & \cdots & 0 \\ \delta_1(f) & \delta_0(f) & 0 & \cdots & 0 \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & 0 \\ \delta_k(f) & \delta_{k-1}(f) & & \cdots & \delta_0(f) \end{pmatrix}$$

where $\delta_i(f) = f^{(i)}(0)/i!$. Since D is a continuous strong ϵ-almost derivation there exist continuous linear functionals $\theta_0, \theta_1, ..., \theta_k$ on $C^n[0,1]$ such that

$$D(f) = \sum_{i=0}^{k} \theta_i(f)\rho(z)^i m_0, \quad f \in C^n[0,1].$$

Thus there is a constant $M > 0$ such that

$$|\theta_j(fg) - \sum_{i=0}^{j}[\delta_{j-i}(f)\theta_i(g) + \delta_{j-i}(g)\theta_i(f)]| \le \epsilon M \|fg\| \tag{1}$$

for all $f, g \in C^n[0,1]$, $j = 0, 1, ..., k$.

Now we define

$$F(f) = \rho(f')D(z), \quad f \in C^n[0,1].$$

Since $2k+2 \le n$, it is easy to show that F is well defined and a continuous derivation from $C^n[0,1]$ into $\mathcal{M}$. $D(z) = \sum_{i=0}^{k} \theta_i(z)\rho(z)^i m_0$ gives

$$F(f) = \sum_{j=0}^{k}\sum_{i=0}^{j} \delta_i(f')\theta_{j-i}(z)\rho(z)^j m_0.$$

We put

$$F_j(f) = \sum_{i=0}^{j} \delta_i(f')\theta_{j-i}(z), \quad f \in C^n[0,1].$$

For a polynomial $p(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$ $(m \ge 2j+2)$, the formula (1) implies $|\theta_j(1)| \le \epsilon M$ and

$$\begin{aligned} &|\theta_j(\alpha_{2j+2}z^{2j+2} + \cdots + \alpha_m z^m)| \\ &\qquad \le \epsilon M \|\alpha_{2j+2}z^{2j+2} + \cdots + \alpha_m z^m\| \\ &\qquad \le \epsilon M[\|p\| + \|\alpha_0 + \alpha_1 z + \cdots + \alpha_{2j+1}z^{2j+1}\|] \\ &\qquad \le 2^{n+1}\epsilon M \|p\|. \end{aligned} \tag{2}$$

Now we prove the following formula by induction ;

$$|\theta_j(z^i) - i\theta_{j-i+1}(z)| \le \epsilon M(2^{i+1} - 1), \quad i = 1, 2, ..., j+1. \tag{3}$$

If $j = 0$, it is trivial. Assume that

$$|\theta_{j-1}(z^i) - i\theta_{j-i}(z)| \le \epsilon M(2^{i+1} - 1),\ j > 1, \quad i = 1, 2, ..., j.$$

From (1) and assumption we obtain for $i = 1, 2, ..., j + 1$,

$$\begin{aligned} |\theta_j(z^i) - i\theta_{j-i+1}(z)| &\le |\theta_j(z^i) - \theta_{j-1}(z^{i-1}) - \theta_{j-i+1}(z)| \\ &+ |\theta_{j-1}(z^{i-1}) - (i-1)\theta_{j-i+1}(z)| \\ &\le \epsilon M(2^{i+1} - 1). \end{aligned}$$

The formula (3) gives

$$\begin{aligned} &|\alpha_2\theta_j(z^2) + \cdots + \alpha_{j+1}\theta_j(z^{j+1}) \\ (4) \qquad &\quad - 2\alpha_2\theta_{j-1}(z) - \cdots - (j+1)\alpha_{j+1}\theta_0(z)| \\ &\quad \le 2^{j+3}\epsilon M\|p\|. \end{aligned}$$

We also show the following formula by induction ;

$$(5) \qquad |\theta_k(z^{j+1})| \le \epsilon M \sum_{i=0}^{k} 2^{j+1-i}, \quad k = 0, 1, 2, ..., j - 1.$$

If $j = 1$ the formula (1) implies $|\theta_0(z^2)| \le 4M\epsilon$. Assume that

$$|\theta_k(z^j)| \le \epsilon M \sum_{i=0}^{k} 2^{j-i}, \quad k = 0, 1, ..., j - 2.$$

If $j \ge 2k + 1$ it follows from (1) that $|\theta_k(z^{j+1})| \le 2^{j+1}\epsilon M$. Otherwise (1) implies

$$|\theta_k(z^{j+1}) - \theta_{2k-j}(z^{k+1})| \le 2^{j+1}\epsilon M.$$

Since $2k - j \le k - 1$ the assumption gives

$$|\theta_{2k-j}(z^{k+1})| \le \epsilon M \sum_{i=0}^{2k-j} 2^{k+1-i}$$

and so

$$|\theta_k(z^{j+1})| \le 2^{j+1}\epsilon M + |\theta_{2k-j}(z^{k+1})| \le \epsilon M \sum_{i=0}^{k} 2^{j+1-i}.$$

Now (1) and (5) give us

$$\begin{aligned} &|\theta_j(\alpha_{j+2}z^{j+2} + \cdots + \alpha_{2j+1}z^{2j+1})| \\ (6) \qquad &\le \epsilon M\|\alpha_{j+2}z^{j+2} + \cdots + \alpha_{2j+1}z^{2j+1}\| \\ &+ \|p\|(|\theta_0(z^{j+1})| + \cdots + |\theta_{j-1}(z^{j+1})|) \\ &\le (2^{2j+2} + j2^{j+2})\epsilon M\|p\|. \end{aligned}$$

The formulas (2), (4) and (6) imply

$$|\theta_j(p) - F_j(p)| \le 2^{n+2}\epsilon M\|p\|.$$

Since θ_j and F_j are continuous, we have

$$|\theta_j(f) - F_j(f)| \le 2^{n+2}\epsilon M\|f\|, \quad f \in C^n[0,1].$$

Thus there exist a constant $\epsilon' > 0$ and a continuous derivation F such that

$$\|D(f) - F(f)\| \le \epsilon'\|f\|, \quad f \in C^n[0,1]$$

where $\epsilon' \to 0$ as $\epsilon \to 0$. This completes the proof of the theorem.

We now consider that an ϵ-almost derivation from $C^n[0,1]$ into a Banach $C^n[0,1]$-module $\mathcal{M}$ is near a derivation.

THEOREM 5. *Let $\mathcal{M}$ be a Banach $C^n[0,1]$-module. If $D : C^n[0,1] \to \mathcal{M}$ is a continuous ϵ- almost derivation and $\rho(z)^i D(z^j) = 0$ for $i + j \ge n+1$, $i,j = 0,1,\ldots,n$ then there is a continuous derivation $F : C^n[0,1] \to \mathcal{M}$ such that*

$$\|D(f) - F(f)\| \le \epsilon'\|f\|, \quad f \in C^n[0,1]$$

where $\epsilon' \to 0$ *as* $\epsilon \to 0$.

Proof. We define $\theta : C^n[0,1] \times C^n[0,1] \to \mathcal{M}$ by $\theta(f,g) = D(fg) - \rho(f)D(g) - \rho(g)D(f)$, $\quad f, g \in C^n[0,1]$. Then θ is a continuous bilinear map. We prove the following formula by induction ; For $m \geq 2$

$$D(z^m) = m\rho(z^{m-1})D(z) + \rho(z^{m-2})\theta(z,z) + \rho(z^{m-3})\theta(z,z^2) + \cdots + \theta(z, z^{m-1}).$$

It is trivial for $m = 2$. Assume that it holds for $m-1$. Then

$$\begin{aligned} D(z^m) &= \theta(z, z^{m-1}) + \rho(z)D(z^{m-1}) + \rho(z^{m-1})D(z) \\ &= m\rho(z^{m-1})D(z) + \rho(z^{m-2})\theta(z,z) + \rho(z^{m-3})\theta(z,z^2) \\ &+ \cdots + \theta(z, z^{m-1}). \end{aligned}$$

Since $\rho(z)^i D(z^j) = 0$ for $i + j \geq n+1$, $i, j = 0, 1, ..., n$, it is easy to show that $\rho(z)^i\theta(z, z^j) = 0$ for $i + j \geq n$. If $n \geq 2$ we get for $p(z) = \alpha_0 + \alpha_1 z + \cdots + \alpha_m z^m$ $(m \geq n)$,

$$\begin{aligned} D(p) &= \rho(p')D(z) + \alpha_n[\rho(z^{n-2})\theta(z,z) + \rho(z^{n-3})\theta(z,z^2) + \cdots \\ &+ \rho(z)\theta(z, z^{n-2}) + \theta(z, z^{n-1})] \\ &+ \alpha_{n-1}[\rho(z^{n-3})\theta(z,z) + \rho(z^{n-4})\theta(z,z^2) + \cdots + \theta(z, z^{n-2})] \\ &+ \cdots + \alpha_2\theta(z,z) + D(\alpha_0) \end{aligned}$$

Since $|\alpha_i| \leq \|p\|$, $i = 0, 1, ..., n$ and $\|\rho\| \geq 1$

(7) $$\|D(p) - \rho(p')D(z)\| \leq \epsilon[(n-1)n/2 + 1]2^n\|\rho\|\|p\|.$$

If $n = 1$ we have $D(p) = D(\alpha_0) + \alpha_1 D(z)$. Thus the formula (7) holds for $n = 1$. By assumption we get $D(z) \in \mathcal{W}$ and, so it follows from Theorem 3 that there exists a unique continuous homomorphism $\gamma : C^{n-1}[0,1] \to \mathcal{B}(\mathcal{W})$ such that

$$\gamma(p)D(z) = \rho(p)D(z), \quad p \in \mathcal{P}.$$

We define $F : C^n[0,1] \to \mathcal{M}$ by $F(f) = \gamma(f')D(z)$. Then F is a continuous derivation which satisfies

$$\|D(f) - F(f)\| \leq \epsilon[(n-1)n/2 + 1]2^n\|\rho\|\|f\|, \quad f \in C^n[0,1].$$

This completes the proof of the theorem.

REMARK. Let $D : C^n[0,1] \to \mathcal{B}(\mathcal{C})$ be the continuous ϵ-almost derivation as in Example. Then $\rho(z)m = 0$ for $m \in \mathcal{C}$ and $D(z^2) = 0$.

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Kil-Woung Jun
Department of Mathematics, Chungnam National University, Taejon 305-764, Korea

Dal-Won Park
Department of Mathematics Education, Kongju National University, Kongju 314-701, Korea