Bull. Korean Math. Soc. **33** (1996), No. 3, pp. 329–334

A NOTE ON MEAN VALUE PROPERTY AND MONOTONICITY

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1. Introduction and Definitions

The notion of approximate derivative was introduced by Denjoy in 1916 [3]. Khintchine [5] proved that Rolle's theorem holds for approximate derivatives and Tolstoff [8] proved that every approximate derivative is of Baire class 1 and has Darboux property. Goffman and Neugebauer [4] proved the above results of Tolstoff [8] in a different and simplified method. Also they [4] proved indirectly (via Darboux property) that approximate derivatives possesss mean value property. The theorems of Goffman and Neugebauer [4] can be stated as follows :

THEOREM A. *Assume that* $f : [0,1] \to R$ *has an approximate derivative* f'_{ap} *everywhere on* $[0,1]$. *Then* f'_{ap} *possesses Darboux property.*

THEOREM B. *Let* $f : [0,1] \to R$ *have an approximate derivative* f'_{ap} *everywhere on* $[0,1]$. *Then Darboux property and mean value property are equivalent for* f'_{ap}.

The purpose of this note is to prove the mean value theorem for approximate derivatives under weaker hypotheses in a direct and simpler method. We also avoid Zorn's lemma as was used by Goffman and Neugebauer [4]. The key step of our proof is the use of a result on the approximate extremum due to O'Malley [6]. As a second application of this result of O'Malley we prove a theorem on monotonicity of functions which improves a result of [4].

Received March 3, 1995.

1991 AMS Subject Classification: 26A24, 26A48.

Key words and phrases: Approximately continuous function, Approximate maximum or minimum, Approximate derivative.

When we call a set or a function to be measurable, we mean it is so in Lebesgue sense. Since every approximately continuous function on an interval is measurable {cf. p. 19[1]}, our purpose will be served if throughout the note we consider only measurable functions f, ϕ etc. defined on $E = [0,1]$. Also for a set A we denote by CA the complement of A.

DEFINITION 1. {*cf.* [2]} The upper right approximate limit of f at ξ, denoted by $u^+(f,\xi)$ or simply $u^+(\xi)$, is the infimum of the numbers K for which the set $E[f > K, x > \xi]$ has zero density at ξ.

DEFINITION 2. {*cf.* [2]} The lower right approximate limit of f at ξ, denoted by $\ell^+(f,\xi)$ or simply $\ell^+(\xi)$, is the supremum of the numbers K for which the set $E[f < K, x > \xi]$ has zero density at ξ.

The left approximate extreme limits are defined likewise.

DEFINITION 3. {*cf.* [2]} The upper right approximate limit of $\frac{f(x)-f(\xi)}{x-\xi}$ at ξ is called upper right approximate derivative of f at ξ and is denoted by ${}_{ap}D^+f(\xi)$.

The other extreme derivatives are defined similarly and denoted by ${}_{ap}D_+f(\xi), {}_{ap}D^-f(\xi), {}_{ap}D_-f(\xi)$. When all the four extreme derivatives are equal at a point ξ, the common value $f'_{ap}(\xi)$ is called the approximate derivative of f at ξ.

DEFINITION 4. {*cf.* [6]} The function f is said to have an approximate maximum at $x_0 \in E$ if $E[f > f(x_0)]$ has density zero at x_0.

An approximate minimum is defined similarly.

2. Lemmas

In this section we present some lemmas which will be required in the next section.

LEMMA 1. $u^+(\xi) = \inf\{\limsup_{x\to\xi+, x\in A} f(x) : A \subset E$ *is measurable and* $d(A,\xi) = 1\}$.

Proof. Let $U^+(\xi)$ denote the right hand side. Now we consider the following cases.

Case I. $-\infty < U^+(\xi) < \infty$.

Let $\varepsilon(>0)$ be arbitrary. Then there exists a measurable set $A \subset E$ with $d(A,\xi) = 1$ such that $\limsup_{x\to\xi+, x\in A} f(x) < U^+(\xi) + \varepsilon$. So there exists a $\delta(>0)$ such that $f(x) < U^+(\xi)+\varepsilon$ for all $x \in A \cap (\xi, \xi+\delta)$. Therefore,

$$\begin{aligned} A[f > U^+(\xi) + \varepsilon, x > \xi] &\subset C[A \cap (\xi, \xi + \delta)] \cap (\xi, \infty) \\ &= [CA \cap (\xi, \infty)] \cup [\xi + \delta, \infty). \end{aligned}$$

Since

$$\begin{aligned} &E[f > U^+(\xi) + \varepsilon, x > \xi] \\ &= A[f > U^+(\xi) + \varepsilon,\ x > \xi] \cup \{E[f > U^+(\xi) + \varepsilon, x > \xi] \cap CA\} \\ &\subset [CA \cap (\xi, \infty)] \cup CA \cup [\xi + \delta, \infty) \\ &= CA \cup [\xi + \delta, \infty), \end{aligned}$$

the density of $E[f > U^+(\xi) + \varepsilon, x > \xi]$ at ξ is zero. So $u^+(\xi) \le U^+(\xi) + \varepsilon$ and hence

$$u^+(\xi) \le U^+(\xi). \tag{1}$$

Let K be a real number such that E $[f > K, x > \xi]$ has density zero at ξ. Let $F = C\{E[f > K, x > \xi]\} \cap E$. Then $d(F,\xi) = 1$ and for all $x \in F, f(x) \le K$. So $U^+(\xi) \le \limsup_{x\to\xi+, x\in F} f(x) \le K$ and since K is arbitrary it follows that

$$U^+(\xi) \le u^+(\xi). \tag{2}$$

In this case the result follows from (1) and (2) .

Case II. $U^+(\xi) = +\infty$.

If possible , let $u^+(\xi) < +\infty$. Then there exists $K < +\infty$ suth that $E[f > K, x > \xi]$ has density zero at ξ. Let $F = C\{E[f > K, x > \xi]\} \cap E$. Then $d(F,\xi) = 1$ and $\limsup_{x\to\xi+, x\in F} f(x) \le K$ so that $U^+(\xi) \le K < +\infty$, a contradiction. So $u^+(\xi) = +\infty$.

Case III. $U^+(\xi) = -\infty$.

Then for arbitrary $M(>0)$ there exists a measurable set $A \subset E$ with $d(A,\xi) = 1$ such that $\limsup_{x\to\xi+, x\in A} f(x) < -M$. So there exists a $\delta(>0)$ such that $f(x) < -M$ for all $x \in A \cap (\xi, \xi+\delta)$. Since $A[f > -M, x > \xi] \subset [CA \cap (\xi,\infty)] \cup [\xi+\delta,\infty)$, it follows that $E[f > -M, x > \xi] \subset CA \cup [\xi+\delta,\infty)$ so that the density of $E[f > -M, x > \xi]$ at ξ is zero. Therefore, $u^+(\xi) \le -M$ which implies $u^+(\xi) = -\infty$.

From the above analysis the following cases are clear.

Case IV. If $u^+(\xi) = \infty$ then $U^+(\xi) = \infty$.

Case V. If $u^+(\xi) = -\infty$ then $U^+(\xi) = -\infty$.

Case VI. If $-\infty < u^+(\xi) < \infty$ then $-\infty < U^+(\xi) < \infty$. and $u^+(\xi) = U^+(\xi)$.

This proves the lemma.

LEMMA 2. $\ell^+(\xi) = \sup\{\liminf_{x\to\xi+, x\in A} f(x) : A \subset E$ *is measurable and* $d(A,\xi) = 1\}$. *The proof is omitted.*

REMARK 1. Similar results are true for left hand extreme approximate limits.

LEMMA 3. {*cf.* Remark 2 [6]}. *If f is approximately continuous and not monotone on $[a,b] \subset E$ then there exists $x_0, a < x_0 < b$, at which f has an approximate maximum or minimum.*

3. Theorems

THEOREM 1. *Let f be approximately continuous on E and ${}_{ap}D^+f = {}_{ap}D^-f, {}_{ap}D_+f = {}_{ap}D_-f$ at every point of E. Then for each pair of points α, β with $0 \le \alpha < \beta \le 1$ there exists a point γ, $\alpha < \gamma < \beta$, such that $f'_{ap}(\gamma) = \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$.*

Proof. Let $\phi(x) = f(x) - f(\alpha) - \frac{x-\alpha}{\beta-\alpha}\{f(\beta) - f(\alpha)\}$. Then ϕ is approximately continuous on E and $\phi(\alpha) = \phi(\beta) = 0$. If ϕ is monotone on $[\alpha,\beta]$ then $\phi \equiv 0$ on $[\alpha,\beta]$ and so ϕ'_{ap} exists everywhere in (α,β) and the theorem follows easily. So we suppose that ϕ is not monotone on $[\alpha,\beta]$. Then by Lemma 3 there exist a point $\gamma, \alpha < \gamma < \beta$, at which

ϕ has an approximate maximum or minimum. We suppose that ϕ has an approximate maximum at γ because the other case is similar.

Since ϕ has an approximate maximum at γ, the set $A = E[\phi \leq \phi(\gamma)]$ has density 1 at γ. Then $\limsup_{x \to \gamma+, x \in A} \frac{\phi(x)-\phi(\gamma)}{x-\gamma} \leq 0$ and so by Lemma 1 ${}_{ap}D^{+}\phi(\gamma) \leq 0$. Also we see that $\liminf_{x \to \gamma-, x \in A} \frac{\phi(x)-\phi(\gamma)}{x-\gamma} \geq 0$ so that ${}_{ap}D^{-}\phi(\gamma) \geq {}_{ap}D_{-}\phi(\gamma) \geq 0$. Since ${}_{ap}D^{+}\phi = {}_{ap}D^{+}f - \frac{f(\beta)-f(\alpha)}{\beta-\alpha}$ etc., it follows from above and the given condition that ${}_{ap}D^{+}\phi(\gamma) = {}_{ap}D_{+}\phi(\gamma) = {}_{ap}D^{-}\phi(\gamma) = {}_{ap}D_{-}\phi(\gamma) = 0$ so that $\phi'_{ap}(\gamma)$ exists and $\phi'_{ap}(\gamma) = 0$ from which the theorem follows . This proves the theorem.

REMARK 2. Under the assumptions of Theorem 1 approximate derivative of f exists on an everywhere dense subset of E.

REMARK 3. If we choose $f(\alpha) = f(\beta)$, a generalization of Rolle's theorem follows from Theorem 1.

THEOREM 2. *If f is approximately continuous and ${}_{ap}D^{-}f \geq 0$, ${}_{ap}D^{+}f \geq 0$ on E, then f is monotone increasing and so continuous on E.*

Proof. First we suppose that ${}_{ap}D^{-}f > 0$ and ${}_{ap}D^{+}f > 0$ on E. If possible suppose that f is not monotone on E. Then by Lemma 3 there exists $\xi, 0 < \xi < 1$, such that f has an approximate maximum or minimum at ξ.

If f has an approximate maximum at ξ, the set $A = E[f \leq f(\xi)]$ has unit density at ξ and if f has an approximate minimum at ξ , the set $B = E[f \geq f(\xi)]$ has unit density at ξ .

Since $\limsup_{x \to \xi+, x \in A} \frac{f(x)-f(\xi)}{x-\xi} \leq 0$ and $\limsup_{x \to \xi-, x \in B} \frac{f(x)-f(\xi)}{x-\xi} \leq 0$, by Lemma 1 and Remark 1 either ${}_{ap}D^{+}f(\xi) \leq 0$ or ${}_{ap}D^{-}f(\xi) \leq 0$, a contradiction. So f is monotone on E. If f is monotone decreasing on E then ${}_{ap}D^{+}f \leq D^{+}f \leq 0$ {cf. p. 219 [7]} which is again a contradiction. Therefore, f is monotone increasing on E.

Now we suppose that ${}_{ap}D^{+}f \geq 0$ and ${}_{ap}D^{-}f \geq 0$ and we choose $\psi(x) = f(x) + \varepsilon x$, where $\varepsilon(> 0)$ is arbitrary. Then ${}_{ap}D^{+}\psi > 0$ and ${}_{ap}D^{-}\psi > 0$ on E so that ψ is monotone increasing on E. Since $\varepsilon(> 0)$ is arbitrary, f is also monotone increasing on E. This proves the theorem.

ACKNOWLEDGEMEMT. The author is thankful to Prof. B. K. Lahiri, University of Kalyani, for pointing out the problem to him.

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