Bull. Korean Math. Soc. **33** (1996), No. 2, pp. 319–328

A HOPF BIFURCATION IN A DOUBLE FREE BOUNDARY PROBLEM WITH PUSHCHINO DYNAMICS

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1. Introduction

In [3], they deal with the free boundary problem with Pushchino dynamics. They showed the existence of solutions and the occurence of a Hopf bifurcation. In this paper, we shall show a Hopf bifurcation occurs for the double free boundaries which is given by (1)(see in [4], [5])

$$
(1)\qquad \begin{cases} v_t = v_{xx} - (c_1 + b)v + c_1 H(x - s(t)) - c_1 H(x - p(t)) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for}\,(x,t) \in \Omega^- \cup \Omega^+, \\ v_x(0,t) = 0 = v_x(1,t) \quad \text{for } t > 0, \\ v(x,0) = v_0(x) \quad \text{for } 0 \le x \le 1, \\ \tau \frac{ds}{dt} = C(v(s(t),t)) \quad \text{for } t > 0, \\ \tau \frac{dp}{dt} = -C(v(p(t),t)) \quad \text{for } t > 0, \\ s(0) = s_0,\ 0 < s_0 < 1, \quad p(0) = p_0,\ 0 < p_0 < 1, \end{cases}
$$

where $v(x,t)$ and $v_x(x,t)$ are assumed continuous in $\Omega = (0,1) \times (0,\infty)$. Here, $H(\cdot)$ is the Heaviside function, $\Omega^- = \{(x,t) \in \Omega : 0 < x < s(t), p(t) < x < 1\}$ and $\Omega^+ = \{(x,t) \in \Omega : s(t) < x < p(t)\}$. The velocity of the interface, $C(v)$, in (1), which specifies the evolution of the interface $s(t)$ and $p(t)$, is determined from the first equation in (1)

Received September 7, 1995.

1991 AMS Subject Classification: Primary 35R35, 35B32; Secondary 35B25, 35K22, 35K57, 58F14, 58F22.

Key words and phrases: free boundary, Hopf bifurcation, Fourier transformation.

*This paper was supported by the research fund of Kyonggi University, 1995.

using asymptotic techniques (see in [1]). The function $C(v)$ can be calculated explicitly as

$$C(v) = \frac{2v - \frac{c_1 - 2a}{c_1 + c_2}}{\sqrt{\left(\frac{c_1 - a}{c_1 + c_2} - v\right)\left(v + \frac{a}{c_1 + c_2}\right)}}$$

where $-c_1 < b < \frac{c_1(c_2 - a)}{c_1 + a}$ and c_1, c_2 are positive constants.

2. The preliminarly results

The existence and uniqueness of solution of (1) was investigated in [2] by using the semigroup theory. We recall the few things from [2]:

Let $G(x,y)$ be Green's function of the operator $A := -\frac{d^2}{dx^2} + (c_1 + b)$ and the domain of the operator A, $D(A) = \{v \in H^{2,2}(0,1) : v_x(0) = v_x(1) = 0\}$. Define a function

$$g(x,s,p) = c_1 \int_0^1 G(x,y)(H(y-s) - H(y-p))dy.$$

Setting $\gamma(s,p) = g(s,s,p)$ and $\eta(s,p) = g(p,s,p)$. We obtain the regular problem of (1) by using the transformation $u(x,t) = v(x,t) - g(x,s(t),p(t))$:

$$(2) \qquad \begin{cases} \frac{d}{dt}(u,s,p) + \tilde{A}(u,s,p) = \frac{c_1}{\tau} f(u,s,p) \\ (u,s,p)(0) = (u_0, s_0, p_0). \end{cases}$$

The operator $\tilde{A}$ is a 3×3 matrix whose the entry of the first row and column is the operator A and the rest terms are all zero. The nonlinear term $f(u,s,p)$ is represented by

$$f(u,s,p) = \begin{pmatrix} G(x,s)C(u(s) + \gamma(s,p)) + G(x,p)C(u(p) + \eta(s,p)) \\ C(u(s) + \gamma(s,p)) \\ -C(u(p) + \eta(s,p)) \end{pmatrix}.$$

In order to show the occurence of a Hopf bifurcation, we need to examine the behavior of the eigenvalues for the linearized problem at the stationary solutions of (2). Thus we shall show the stationary solution of (1) (or (2)) exists and the Hopf bifurcation occurs in the next section.

3. The Hopf bifurcation

3.1 The stationary solutions

Let $u(x,t) = u^*(x)$, $s(t) = s^*$, $p(t) = p^*$ and the time derivatives in (2) equal to zero we obtain the stationary problem:

$$\begin{aligned} Au^* &= \frac{c_1}{\tau} C(u^*(s^*) + \gamma(s^*, p^*)) \cdot G(\cdot, s^*) \\ &\quad + \frac{c_1}{\tau} C(u^*(p^*) + \eta(s^*, p^*)) \cdot G(\cdot, p^*) \\ 0 &= \frac{1}{\tau} C(u^*(s^*) + \gamma(s^*, p^*)) \\ 0 &= -\frac{1}{\tau} C(u^*(p^*) + \eta(s^*, p^*)). \end{aligned}$$

For nonzero τ we obtain the following theorem:

THEOREM 1. *If* $0 < \dfrac{c_1 - 2a}{2(c_1 + c_2)} < \dfrac{1}{c_1 + b}$ *then (2) has a unique stationary solution* $(u^*(x), s^*, p^*) = (0, s^*, p^*)$ *for all* $0 < \tau < \infty$ *with* $p^* = 1 - s^*$, $s^* \in (0, 1/2)$. *The linearization of* f *at* $(0, s^*, p^*)$ *is*

$$\begin{aligned} &Df(0, s^*, p^*)(\hat{u}, \hat{s}, \hat{p}) \\ =&\frac{a + c_1}{c_1} \Big(\hat{u}(s^*) + \gamma_s(s^*, p^*)\hat{s} + \gamma_p(s^*, p^*)\hat{p}\Big) \cdot \Big(G(s^*, p^*), 1, 0\Big) \\ &+ \frac{a + c_1}{c_1} \Big(\hat{u}(p^*) + \eta_s(s^*, p^*)\hat{s} + \eta_p(s^*, p^*)\hat{p}\Big) \cdot \Big(G(s^*, p^*), 0, -1\Big). \end{aligned}$$

The pair $(0, s^*, p^*)$ *corresponds to a unique steady state* (v^*, s^*, p^*) *of (1) for* $\tau \neq 0$ *with* $v^*(x) = g(x, s^*, p^*)$.

Proof. We rewrite $\gamma(s,p)$ and $\eta(s,p)$ as

$$\begin{aligned} &\gamma(s, p) = g(s, s, p) \\ &= \eta(s, p) + \frac{\sinh \sqrt{c_1 + b}(1 - s - p)}{(c_1 + b) \sinh \sqrt{c_1 + b}} \Big(\cosh(\sqrt{c_1 + b}\,(s - p)) - 1 \Big). \end{aligned}$$

Since $C(\cdot) = 0$, we have $\gamma(s,p) = \eta(s,p) = \frac{c_1 - 2a}{2(c_1 + c_2)}$. From this, we obtain $\sinh \sqrt{c_1 + b}(1 - s - p) = 0$. Thus we have $C(r) = 0$ iff $s + p =$

1. Therefore we only need to show the existence of s^* which satisfies $\gamma(s^*, 1-s^*) = \frac{c_1-2a}{2(c_1+c_2)}$ for $s^* \in (0,1)$. Now, we define

$$\begin{aligned}\Gamma(s) &:= \gamma(s, 1-s) - \frac{c_1-2a}{2(c_1+c_2)} \\ &= \frac{\sinh\sqrt{c_1+b} - \sinh\sqrt{c_1+b}(2s-1) - \sinh 2\sqrt{c_1+b}\,s}{2(c_1+b)\sinh\sqrt{c_1+b}} - \frac{c_1-2a}{2(c_1+c_2)}.\end{aligned}$$

Then $\Gamma(s)$ is solvable with $s \in (0, 1/2)$, because $\Gamma'(s) < 0$, $\Gamma(0) = \frac{1}{(c_1+b)} - \frac{c_1-2a}{2(c_1+c_2)} > 0$ and $\Gamma(1/2) < 0$. The formula for $Df(0, s^*, p^*)$ follows from the differentiation and the relation $C'(\frac{c_1-2a}{2(c_1+c_2)}) = \frac{c_1+c_2}{c_1}$. Using Theorem 4 in [2], we obtain the corresponding steady state (v^*, s^*, p^*) for (1). □

3.2 A Hopf bifurcation

We now show that a Hopf bifurcation occurs as the new parameter $\mu, \mu = \dfrac{c_1+c_2}{\tau}$ varies. The linearized eigenvalue problem of (2) is given by

$$(-\tilde{A} + \mu Df(0, s^*, p^*))(u, s, p) = \lambda(u, s, p)$$

which is equivalent to

$$\begin{aligned}(3)\qquad Au + \lambda u &= \mu(\gamma_s(s^*, p^*)s + \gamma_p(s^*, p^*)p + u(s^*))G(x, s^*) \\ &\quad + \mu(\eta_s(s^*, p^*)s + \eta_p(s^*, p^*)p + u(p^*))G(x, p^*) \\ (4)\qquad \lambda s &= \mu(\gamma_s(s^*, p^*)s + \gamma_p(s^*, p^*)p + u(s^*)) \\ (5)\qquad \lambda p &= -\mu(\eta_s(s^*, p^*)s + \eta_p(s^*, p^*)p + u(p^*)).\end{aligned}$$

We have the following lemma:

LEMMA 2. *For $\mu^* \in \mathbf{R}\backslash\{0\}$, there exists a C^1-curve $\mu \to (\phi(\mu), \lambda(\mu))$ of eigendata such that $\phi(\mu^*) = \phi^*$ and $\lambda(\mu^*) = i\beta$ where ϕ^* is an eigenfunction of $-\tilde{A} + \mu^* Df(0, s^*, p^*)$ with eigenvalue $i\beta$.*

Proof. Let $\phi^* = (\psi_0, s_0, p_0) \in D(A) \times \mathbf{R}^2$. First, we see that $s_0 \neq 0$ and $p_0 \neq 0$, for otherwise, by (3), $(A + i\beta)\psi_0 = i\beta G(\cdot, s^*)s_0 -$

$i\beta G(\cdot, p^*)p_0 = 0$, which is not possible because A is symmetric. So without loss of generality, let $s_0 = 1$ and $p_0 = 1$. Then by (3) $E(\psi_0, i\beta, \mu^*) = 0$, where

$$E : D(A) \times \mathbf{C} \times \mathbf{R} \times \mathbf{R} \longrightarrow X_{\mathbf{C}} \times \mathbf{C} \times \mathbf{C},$$

$$E(u, \lambda, \mu)$$

$$= \begin{pmatrix} (A+\lambda)u - \mu \cdot (\gamma_s(s^*, p^*) + \gamma_p(s^*, p^*) + u(s^*))G(\cdot, s^*) \\ -\mu(\eta_s(s^*, p^*) + \eta_p(s^*, p^*) + u(p^*))G(\cdot, p^*) \\ \\ \lambda - \mu \cdot (\gamma_s(s^*, p^*) + \gamma_p(s^*, p^*) + u(s^*)) \\ \\ \lambda + \mu \cdot (\eta_s(s^*, p^*) + \eta_p(s^*, p^*) + u(p^*)) \end{pmatrix}.$$

The equation $E(u, \lambda, \mu) = 0$ is equivalent that λ is an eigenvalue of $-\widetilde{A} + \mu Df(0, s^*, p^*)$ with eigenfunction $(u, 1, 1)$. We want to apply the implicit function theorem to E, and therefore have to check that E is in C^1 and that

(6)

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu_0) : D(A) \times \mathbf{C} \to L^2(0,1) \times \mathbf{C}) \text{ is an isomorphism.}$$

Now it is easy to see that E is in C^1. The mapping

$$D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)(\hat{u}, \hat{\lambda})$$

$$= \begin{pmatrix} (A + i\beta)\hat{u} - \mu^*\hat{u}(s^*) \cdot G(\cdot, s^*) - \mu^*\hat{u}(p^*) \cdot G(\cdot, p^*) + \hat{\lambda}\psi_0 \\ -\mu^*\hat{u}(s^*) + \hat{\lambda} \\ \mu^*\hat{u}(p^*) + \hat{\lambda} \end{pmatrix}$$

is a compact perturbation of the mapping

$$(\hat{u}, \hat{\lambda}) \longmapsto \Big((A + i\beta)\hat{u}, \hat{\lambda}\Big)$$

which is invertible. As a consequence, $D_{(u,\lambda)}E(\psi_0, i\beta, \mu^*)$ is a Fredholm operator of index 0. Thus to verify (6), it suffices to show that the system

$$(7) \qquad \begin{cases} (A + i\beta)\hat{u} + \hat{\lambda}\psi_0 = \mu^*(\hat{u}(s^*)G(\cdot, s^*) + u(p^*)G(\cdot, p^*)) \\ \hat{\lambda} = \mu^*\hat{u}(s^*) \\ \hat{\lambda} = -\mu^*\hat{u}(p^*) \end{cases}$$

necessarily implies that $\hat{u} = 0$, $\hat{\lambda} = 0$. Thus let $(\hat{u}, \hat{\lambda})$ be a solution of (7), and define $\psi_1 := \psi_0 - G(\cdot, s^*) + G(\cdot, p^*)$. Then

$$(A + i\beta)\hat{u} + \hat{\lambda}\psi_1 = 0 . \tag{8}$$

Also, ψ_1 is a solution to the equation

$$(A + i\beta)\psi_1 = -\delta_{s^*} + \delta_{p^*} \tag{9}$$

$$i\beta = \mu^* \cdot \Big(\gamma_s(s^*, p^*) + \gamma_p(s^*, p^*) + \psi_1(s^*) + G(s^*, s^*)\Big) \tag{10}$$

$$i\beta = -\mu^* \cdot \Big(\eta_s(s^*, p^*) + \eta_p(s^*, p^*) + \psi_1(p^*) + G(s^*, p^*)\Big). \tag{11}$$

From the equation (9), we have

$$\operatorname{Im}(\psi_1(s^*) - \psi_1(p^*)) = \beta \int_0^1 |\psi_1|^2 .$$

If we add (10) and (11), then

$$\mu^* \int_0^1 |\psi_1|^2 = 2 . \tag{12}$$

From (9) we can then calculate $\hat{u}(s^*)$ as $\int_0^1 \psi_1(A+i\beta)\hat{u} = -\hat{u}(s^*) + \hat{u}(p^*)$ which, together with (8), (9) and (12), implies that

$$\hat{\lambda} \int_0^1 \psi_1^2 = \hat{u}(s^*) - \hat{u}(p^*) = 2\hat{\lambda}/\mu^* = \hat{\lambda} \int_0^1 |\psi_1|^2 .$$

As a result

$$\hat{\lambda}\left(\int_0^1 |\psi_1|^2 - \psi_1^2\right) = 0 ,$$

which implies $\hat{\lambda} = 0$, for otherwise $\operatorname{Im}\psi_1 = \operatorname{Im}\psi_0 = 0$, which is a contradiction. So we conclude that $\hat{\lambda} = 0$. And so we have $\hat{u} = 0$.

We have thus shown (6), and get a C^1-curve $\mu \mapsto (\phi(\mu), \lambda(\mu))$ of eigendata such that $\phi(\mu^*) = \phi^*$ and $\lambda(\mu^*) = i\beta$. $\square$

Now we shall use the Fourier cosine transformation to show the transverality condition and uniqueness of μ^*. If we use $v(x,t) = u(x,t) + g(x,s,p)$, the eigenvalue problem is obtained by

$$\lambda v = v_{xx} - (c_1 + b)v - c_1(\delta_{s^*} - \delta_{p^*}) \tag{13}$$

$$\lambda = \mu\big((v^*)'(s^*) + v(s^*)\big) \tag{14}$$

$$\lambda = -\mu\big((v^*)'(p^*) + v(p^*)\big). \tag{15}$$

We take a Fourier cosine transformation in the equation (13), then we have

$$\begin{aligned} v(x) &= -2c_1 \sum_{k=1}^{\infty} \frac{\cos k\pi s^* - \cos k\pi p^*}{(k\pi)^2 + c_1 + b + \lambda} \cos k\pi x \\ &= -4c_1 \sum_{k=1}^{\infty} \frac{\cos(2k-1)\pi s^*}{((2k-1)\pi)^2 + c_1 + b + \lambda} \cos(2k-1)\pi x \end{aligned}$$

since $p^* = 1 - s^*$. Furthermore, by using Green's function

$$v(x) = -G_\lambda(x, s^*) + G_\lambda(x, p^*). \tag{16}$$

Now, we add the equation (14) and (15):

$$\mu\big((v^*)'(s^*) - G_\lambda(s^*, s^*) + G_\lambda(s^*, 1 - s^*)\big) = \lambda \tag{17}$$

since $(v^*)'(s^*) = (v^*)'(1 - s^*)$. Here is the main theorem.

THEOREM 3. *For a given pure imaginary eigenvalue $i\beta$, $\beta \neq 0$, there exists a unique μ^* such that $(0, s^*, p^*, \mu^*)$ is a Hopf point.*

Proof. We assume that $\beta > 0$ and let $\lambda = i\beta$ in (17), then the real and imagnary parts are obtained by

$$\mu \operatorname{Im}\big((-G_\beta(s^*, s^*) + G_\beta(s^*, 1 - s^*)\big) = \beta \tag{18}$$

$$\mu\big((v^*)'(s^*) + \operatorname{Re}(-G_\beta(s^*, s^*) + G_\beta(s^*, 1 - s^*))\big) = 0 \tag{19}$$

where G_β is Green's function of the operator $A + i\beta$. If we know the existence of β in (19), we may find the value of μ^* corresponding β in (18). Thus, we define

$$T(\beta) = (v^*)'(s^*) + \mathrm{Re}(-G_\beta(s^*, s^*) + G_\beta(s^*, 1 - s^*)).$$

Then

$$\begin{aligned} T(0) &= (v^*)'(s^*) + (-G(s^*, s^*) + G(s^*, 1 - s^*)) \\ &= \frac{1}{\sqrt{c_1 + b}\sinh\sqrt{c_1 + b}}\left(1 - \cosh(\sqrt{c_1 + b}(1 - 2s^*))\right) \\ &< 0 \end{aligned}$$

and $\lim_{\beta\to\infty} T(\beta) = (v^*)'(s^*) > 0$. Furthermore, $T'(\beta) > 0$. Therefore there is a unique β such that $T(\beta) = 0$. From this β, the μ can be uniquely determined from (18).

Now we only need to show the transversality condition. Differentiate with respect to μ in (17) then we have

$$\lambda'(\mu)(1/\mu + G'_\lambda(s^*, s^*) - G'_\lambda(s^*, 1 - s^*)) = \frac{\lambda}{\mu^2}.$$

Evaluating at $\mu = \mu^*$ (note $\lambda(\mu^*) = i\beta$),

$$\lambda'(\mu^*)(\frac{1}{\mu^*} + G'_\beta(s^*, s^*) - G'_\beta(s^*, 1 - s^*)) = \frac{i\beta}{(\mu^*)^2}.$$

The real part of $\lambda'(\mu^*)$ is

$$\mathrm{Re}\lambda'(\mu^*) = \frac{\frac{\beta}{(\mu^*)^2}(D - F)}{(C - E + 1/\mu^*)^2 + (D - F)}$$

where $C + iD = G'_\beta(s^*, s^*)$ and $E + iF = G'_\beta(s^*, 1 - s^*)$. We only need to examine the sign of $D - F$. $D = \mathrm{Im}G'_\beta(s^*, s^*)$ and

$$D = \frac{4\beta}{(4 + \beta^2)^2} + 4c_1\beta\sum_{k=1}^{\infty}\frac{(\cos k\pi s^*)^2(k^2\pi^2 + c_1 + b)}{((k^2\pi^2 + c_1 + b)^2 + \beta^2)^2}$$

and

$$F = \frac{4\beta}{(4+\beta^2)^2} + 4c_1\beta \sum_{k=1}^{\infty} \frac{\cos k\pi s^* \cos k\pi(1-s^*)(k^2\pi^2+c_1+b)}{((k^2\pi^2+c_1+b)^2+\beta^2)^2}.$$

Thus,

$$D - F = 8c_1\beta \sum_{k=1}^{\infty} \frac{(\cos(2k-1)\pi s^*)^2((2k-1)^2\pi^2+c_1+b)}{(((2k-1)^2\pi^2+c_1+b)^2+\beta^2)^2}.$$

The transversality condition $\mathrm{Re}\lambda'(\mu^*) > 0$ is satisfied. □

Therefore, we have the following theorem for the Hopf bifurcation of (1):

THEOREM 4. *Assume that* $0 < \frac{c_1 - 2a}{2(c_1+c_2)} < \frac{1}{c_1+b}$ *so that (1), respectively (2), has a unique stationary solution* $(0, s^*, p^*)$ *, respectively* (v^*, s^*, p^*)*, for all* $\mu > 0$ *with* $p^* = 1 - s^*$*. Then there exists a unique* $\mu^* > 0$ *such that the linearization* $-\widetilde{A} + \mu^* Df(0, s^*, p^*)$ *has a purely imaginary pair of eigenvalues. The point* $(0, s^*, p^*, \mu^*)$ *is then a Hopf point for (1) and there exists a* C^1*-curve of nontrivial periodic orbits for (1), (2), respectively, bifurcating from* $(0, s^*, p^*, \mu^*)$*,* (v^*, s^*, p^*, μ^*)*, respectively.*

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