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LINEAR TRANSFORMATIONS THAT PRESERVE THE ASSIGNMENT ON $R = E_m$ AND $S = (s_1, \cdots, s_n)$

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1. Introduction

For positive integral vectors $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$, we consider the class $\mathcal{U}(R, S)$ of all $m \times n$ matrices of 0's and 1's with the row sum vector R and the column sum vector S. Throughout the paper we assume that $\mathcal{U}(R, S) \neq \emptyset$. If $\sum_{i=1}^{m} r_i \neq \sum_{j=1}^{n} s_j$, then $\mathcal{U}(R, S) = \emptyset$. So we assume throughout that $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$ and $0 < r_i, s_j \leq n$ for each i, j where $n \leq m$. We refer to matrices in $\mathcal{U}(R, S)$ as (R, S)-*assignments*, or as *assignments* when R and S are fixed in the discussion.

Let $\overline{\mathcal{U}(R, S)}$ denote the convex hull of the (R, S)-assignment $\mathcal{U}(R, S)$ considered as points in real mn-dimensional space. Since each assignment has all entries equal to 0 or 1(hence is a vector of the mn-dimensional unit cube), it follows readily that the assignments are precisely the vertices(extreme points) of $\overline{\mathcal{U}(R, S)}$. Brualdi, Hartfiel and Hwang [5] proved that $\overline{\mathcal{U}(R, S)}$ is a convex polytope.

Once again let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be positive integral vectors for which $\mathcal{U}(R, S) \neq \emptyset$. Let $X = [x_{ij}]$ be an $m \times n$ matrix. We define the *support* of X to be the set $\text{supp}(X) = \{(i, j) : x_{ij} \neq 0\}$. The $(R, S)-$*assignment function*, or an *assignment function*, $\mathrm{P}_{R,S}(\cdot)$ is now defined by

$$\mathrm{P}_{R,S}(X) = \sum_{A \in \mathcal{U}(R,S)} \prod_{(i,j) \in \text{supp}(A)} x_{ij}. \tag{1.1}$$

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A well-known case of an assignment function occurs when $m = n$ and $R = S = (1, \cdots, 1)$. In this case, $\mathrm{P}_{R,S}(X)$ counts the number of permutation matrices P with $P \leq X$ and hence $\mathrm{P}_{R,S}(X)$ is the permanent of X, $\mathrm{per}(X)$.

In [6], Marcus and May characterized the permanent preservers and P. Botta [4] used another method for the characterization of the linear transformation that preserves the permanent. Marcus, Minc and Moyls [7] characterized the permanent preservers on the space of doubly stochastic matrices. In [1,2], the author characterized the nature of all linear transformations T on M_n into itself such that $\mathrm{P}_{R,S}(T(X)) = \mathrm{P}_{R,S}(X)$ for all $X \in M_n$. For such an assignment preserver T, there exist permutation matrices P, Q and diagonal matrices $D = \mathrm{diag}\{d_1, \cdots, d_n\}$, $L = \mathrm{diag}\{l_1, \cdots, l_n\}$ in M_n such that $\prod_{i=1}^{n} d_i^{r_i} \cdot \prod_{j=1}^{n} l_j^{s_j} = 1$ and either $T(X) = PDXLQ$ for all $X \in M_n$ or $T(X) = PDX^TLQ$ and $R = S$ for all $X \in M_n$.

A nonnegative real matrix is called *doubly stochastic* if all its row sums and column sums equal to 1. The set of all $n \times n$ doubly stochastic matrices is denoted by Ω_n. A vector E_m denote the m-tuple of 1's, i.e., $E_m = (1, \cdots, 1)$. Let $R = E_m$ and $S = (s_1, \cdots, s_n)$ where s_i, $i = 1, \cdots, n$, are positive integers with $s_1 + \cdots + s_n = m$. Then $\overline{\mathcal{U}(R,S)}$ is a generalization of Ω_n and $\mathrm{P}_{R,S}(\cdot)$ is a generalized function of permanent. In the present paper we consider those linear mappings that preserve the assignment functions of $\overline{\mathcal{U}(R,S)}$. By an assignment preserver on $\overline{\mathcal{U}(R,S)}$ we mean a mapping T on $\overline{\mathcal{U}(R,S)}$ into itself such that, for $A, B \in \overline{\mathcal{U}(R,S)}$ and for real numbers α, β, $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$, $\alpha + \beta = 1$,

$$T(\alpha A + \beta B) = \alpha T(A) + \beta T(B), \tag{1.2}$$

$$\mathrm{P}_{R,S}(T(A)) = \mathrm{P}_{R,S}(A). \tag{1.3}$$

We shall show that for such T there exist fixed permutation matrices P, Q such that either

$$T(A) = PAQ \text{ for all } A \in \overline{\mathcal{U}(R,S)}$$

or

$$R = S,\ T(A) = PA^TQ \text{ for all } A \in \overline{\mathcal{U}(R,S)} \text{ and } SQ = S.$$

2. Results

Let $R = E_m$ and $S = (s_1, \cdots, s_n)$ where s_i, $i = 1, \cdots, n$, are positive integers with $s_1 + \cdots + s_n = m$. We consider those linear mappings which preserve the assignment functions of $\overline{\mathcal{U}(R,S)}$.

The following lemma will be useful for our study.

LEMMA 1. *Let $R = (r_1, \cdots, r_m)$ and $S = (s_1, \cdots, s_n)$ be vectors of positive integers with $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$. Let P and Q are permutation matrices. Then*

$$P_{R,S}(X) = P_{R,S}(PXQ) \tag{2.1}$$

if and only if $PR^T = R^T$ and $SQ = S$.

Proof. For each $A \in \mathcal{U}(R,S)$, $PAQ \in \mathcal{U}(R,S)$ only if $PR^t = R^t$ and $SQ = S$. This establishes the necessity. Now, suppose that $PR^t = R^t$ and $SQ = S$. Then,

$$\begin{aligned} \mathrm{P}_{R,S}(X) &= \sum_{A \in \mathcal{U}(R,S)} \prod_{(i,j) \in \mathrm{supp}(A)} x_{ij} \\ &= \sum_{PAQ \in \mathcal{U}(R,S)} \prod_{(i,j) \in \mathrm{supp}(PAQ)} x_{ij} \\ &= \sum_{A \in \mathcal{U}(R,S)} \prod_{(i,j) \in \mathrm{supp}(A)} (PXQ)_{ij} \\ &= \mathrm{P}_{R,S}(PXQ). \qquad \square \end{aligned}$$

REMARK. We note that the assignment is not invariant under permutations of rows or columns, and is not invariant under transposition. For example, if $R = (2,2,2)$ and $S = (3,2,1)$, then $\mathcal{U}(R,S) = \{A_1, A_2, A_3\}$ where

$$A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \; A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

Then $\mathrm{P}_{R,S}(A_1) = 1$. But $\mathrm{P}_{R,S}(A_1^T) = 0$. And let

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

then

$$A_1 Q = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

So, $\mathrm{P}_{R,S}(A_1) = 1$ and $\mathrm{P}_{R,S}(A_1 Q) = 0$.

Note that if $R = S$ then $\mathcal{U}(R,S) = \mathcal{U}(S,R) = \mathcal{U}(R,R)$ and hence, in this case, $\mathrm{P}_{R,S}(A) = \mathrm{P}_{R,S}(A^T)$ for all A. We have thus established the following lemma.

LEMMA 2. *The transpose operator preserves the assignment function $P_{R,S}$ if and only if $R = S$.*

Let $R = E_m$ and let $S = (s_1, \cdots, s_n)$ where s_i, $i = 1, \cdots, n$, are positive integers with $s_1 + \cdots + s_n = m$. Let T be a mapping on $\overline{\mathcal{U}(R,S)}$ into itself satisfying (1.2) and (1.3). In [5, Theorem 4.5], it was shown that for $A \in \overline{\mathcal{U}(R,S)}$, $\mathrm{P}_{R,S}(A) = 1$ if and only if A is an assignment. It follows that T maps $\mathcal{U}(R,S)$ into itself.

Once again let $R = E_m$ and let $S = (s_1, \cdots, s_n)$ where s_i, $i = 1, \cdots, n$, are positive integers with $s_1 + \cdots + s_n = m$. And, define the mapping φ on $\overline{\mathcal{U}(R,S)}$ by

$$\varphi(X) = \Pi T(X) \Sigma \tag{2.2}$$

where Π is a permutation matrix of order m and Σ is a permutation matrix of order n such that $S\Sigma = S$. Then φ has properties (1.2), (1.3) and $\varphi(A) = \Pi T(A) \Sigma$ for $A \in \mathcal{U}(R,S)$. Suppose that $X \in \overline{\mathcal{U}(R,S)}$. Since X is in the convex hull, $X = \sum_{i=1}^k \lambda_i A_i$, $A_i \in \mathcal{U}(R,S)$, $0 \le \lambda_i \le 1$, $i = 1, \cdots, k$ and $\sum_{i=1}^k \lambda_i = 1$ and hence $\varphi(X) = \sum_{i=1}^k \lambda_i \varphi(A_i)$. Since $\varphi(A) = \Pi T(A) \Sigma \in \mathcal{U}(R,S)$ for $A \in \mathcal{U}(R,S)$, it is sufficient to discuss the action of φ on (R,S)-assignments.

Let $\mathcal{B}(V_1, V_2)$ be the set of all bipartite graphs with vertex sets V_1 and V_2 where $|V_1| = m$, $|V_2| = n$. Let G_{ij} be an element of $\mathcal{B}(V_1, V_2)$ whose edge is the only one such that $e_{ij} = \{i, j\}$ for $i \in V_1$, and $j \in V_2$. The mapping L is called a linear mapping of $\mathcal{B}(V_1, V_2)$ into itself if $L(G_1 \cup G_2) = L(G_1) \cup L(G_2)$ for each $G_1, G_2 \in \mathcal{B}(V_1, V_2)$. Define the set $\mathcal{G}_{m,n}$ by

$$\begin{aligned} \mathcal{G}_{m,n} = \{G(V_1, V_2) \in \mathcal{B}(V_1, V_2) :&\deg(i) = 1 \text{ for all } i \in V_1 \\ &\text{and } \deg(j) = s_j \text{ for each } j \in V_2\}. \end{aligned} \tag{2.3}$$

Then the degree sequence of V_1 is E_m and the degree sequence of V_2 is $(s_1, \cdots, s_n)$ such that $\sum_{j=1}^{n} s_j = m$.

LEMMA 3. *If ϑ maps on $\mathcal{U}(R, S)$ into itself such that, for $B \in \mathcal{U}(R, S)$, $\vartheta(B) = C$ for some $C \in \mathcal{U}(R, S)$, then $\vartheta(A) = PAQ$ for $A \in \mathcal{U}(R, S)$ where P is a permutation matrix of order m and Q is a permutation matrix of order n such that $SQ = S$.*

Proof. For any $G, H \in \mathcal{G}_{m,n}$ there exists a graph isomorphism L. A graph isomorphism of a simple graph is a permutation which preserves the adjacency of each vertex.

Let the degree sequence of V_2 is $S = (s_1, \cdots s_n)$ and it is fixed, without loss of generality, we may assume that its permutation is the identity. Now, let L_1 be a graph isomorphism with a permutation σ which preserves the degree of V_1. For any subgraph G_{ij} of G, since $L_1(G_{ij}) = G_{\sigma(i)j}$ where σ is in the symmetric group $\mathcal{S}_m$, $L_1(G_{ij} \cup G_{pq}) = G_{\sigma(i)j} \cup G_{\sigma(p)q} = L_1(G_{ij}) \cup L_1(G_{pq})$, for $i, p \in V_1$, and $j, q \in V_2$. For $G \in \mathcal{G}_{m,n}$, $G = \bigcup_{i=1}^{m} G_{ij}$, $j = 1, \cdots, n$,

$$\begin{aligned} L_1(G) = L_1(\bigcup_{i=1}^{m} G_{ij}) &= \bigcup_{i=1}^{m} L_1(G_{ij}) \\ &= \bigcup_{i=1}^{m} G_{\sigma(i)j}, \quad \sigma \in \mathcal{S}_m. \end{aligned}$$

Since $L_1(G \cup H) = L_1(G) \cup L_1(H)$ for $G, H \in \mathcal{G}_{m,n}$, L_1 is a linear mapping.

Let $A(G)$ be the adjacency matrix of a graph G. Define $\psi : \mathcal{G}_{m,n} \to \mathcal{U}(R,S)$ by $\psi(G) = A(G)$. Then ψ is a bijection. Let P be the permutation matrix with respect to $\sigma \in \mathcal{S}_m$. Then $\psi \circ L_1 \circ \psi^{-1}$ maps on $\mathcal{U}(R,S)$ into itself.

Now, without loss of generality, we may assume that its permutation matrix P is the identity since the degree sequence of V_1 is $R = E_m$. Let L_2 be a graph isomorphism with a permutation τ which preserves the degree of V_2. That is $L_2(G_{ij}) = G_{i\tau(j)}$ for some τ in the symmetric group $\mathcal{S}_n$. Then L_2, which is similar to L_1, is a linear mapping. Let Q be a permutation matrix with respect to $\tau \in \mathcal{S}_n$. Since $s_1, \cdots, s_n$ are arbitrary positive integers such that $s_1 + \cdots + s_n = m$, the permutation matrix Q satisfy $SQ = S$. Then $\psi \circ L_2 \circ \psi^{-1}$ maps on $\mathcal{U}(R,S)$ into itself.

Since ψ is a bijection, for $G \in \mathcal{G}_{m,n}$, there exists $B \in \mathcal{U}(R,S)$ such that $\psi(G) = A(G) = B$. Let $L = L_1 \circ L_2$. Then L is a linear mapping of $\mathcal{G}_{m,n}$ into itself. Let $\vartheta = \psi \circ L \circ \psi^{-1}$. Then ϑ maps on $\mathcal{U}(R,S)$ into itself and, for any $B \in \mathcal{U}(R,S)$,

$$\begin{aligned}\vartheta(B) &= (\psi \circ L \circ \psi^{-1})(B) = \psi(L(\psi^{-1}(B))) \\ &= \psi(L(G)) = A(L(G)) = PA(G)Q = PBQ,\end{aligned}$$

where P and Q are permutation matrices of order m and n, respectively. □

For any $X \in \overline{\mathcal{U}(R,S)}$, $X = \sum_{i=1}^{k} \lambda_i A_i$, for some $A_i \in \mathcal{U}(R,S)$, $0 \le \lambda_i \le 1$, $i = 1, \cdots, k$ and $\sum_{i=1}^{k} \lambda_i = 1$. Since φ is a mapping of $\mathcal{U}(R,S)$ into itself, for permutation matrices P and Q,

$$\begin{aligned}T(X) &= \Pi^T \varphi(X) \Sigma^T \\ &= \Pi^T \sum_{i=1}^{k} \lambda_i \varphi(A_i) \Sigma^T \\ &= \Pi^T \sum_{i=1}^{k} \lambda_i P A_i Q \Sigma^T \\ &= \Pi^T P X Q \Sigma^T.\end{aligned}$$

Now, we have immediately our results:

THEOREM 4. *Let $R = E_m$ and let $S = (s_1, \cdots, s_n)$ where the s_i's are positive integers with $s_1 + \cdots + s_n = m$. Let T be a linear mapping of $\overline{\mathcal{U}(R,S)}$ into itself such that $P_{R,S}(T(X)) = P_{R,S}(X)$ for all $X \in \overline{\mathcal{U}(R,S)}$. Then there exist permutation matrices P and Q such that either*

$$T(X) = PXQ \text{ for all } X \in \overline{\mathcal{U}(R,S)} \tag{2.4}$$

or

$$R = S,\ T(X) = PX^TQ \text{ for all } X \in \overline{\mathcal{U}(R,S)} \text{ and } SQ = S, \tag{2.5}$$

where the permutation matrix Q satisfy $SQ = S$.

Proof. For any $X \in \overline{\mathcal{U}(R,S)}$, $X = \sum_{i=1}^{k} \lambda_i A_i$, for some $A_i \in \mathcal{U}(R,S)$, $0 \le \lambda_i \le 1$, $i = 1, \cdots, k$ and $\sum_{i=1}^{k} \lambda_i = 1$. Since φ is a mapping of $\mathcal{U}(R,S)$ into itself, for permutation matrices P and Q,

$$\begin{aligned} T(X) &= \Pi^T \varphi(X) \Sigma^T \\ &= \Pi^T \sum_{i=1}^{k} \lambda_i \varphi(A_i) \Sigma^T \\ &= \Pi^T \sum_{i=1}^{k} \lambda_i P A_i Q \Sigma^T \\ &= \Pi^T P X Q \Sigma^T. \qquad \square \end{aligned}$$

COROLLARY 5. *Let $R = S = E_n$. Let T be a linear mapping of $\overline{\mathcal{U}(R,S)}$ into itself such that $P_{R,S}(T(X)) = P_{R,S}(X)$ for all $X \in \overline{\mathcal{U}(R,S)}$. Then there exist permutation matrices P and Q such that either*

$$T(X) = PXQ \text{ for all } X \in \overline{\mathcal{U}(R,S)} \tag{2.6}$$

or

$$T(X) = PX^TQ \text{ for all } X \in \overline{\mathcal{U}(R,S)}. \tag{2.7}$$

Proof. If $R = S = E_n$, then $\overline{\mathcal{U}(R,S)}$ is the space of doubly stochastic matrices and $\mathrm{P}_{R,S}(X) = \mathrm{per} X$. $\square$

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