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A DUPLICATION FORMULA FOR THE DOUBLE GAMMA FUNCTION Γ_2

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The double Gamma function had been defined and studied by Barnes [4], [5], [6] and others in about 1900, not appearing in the tables of the most well-known special functions, cited in the exercise by Whittaker and Watson [25, p. 264]. Recently this function has been revived according to the study of determinants of Laplacians [8], [11], [15], [16], [19], [20], [22] and [24]. Shintani [21] also uses this function to prove the classical Kronecker limit formula. Its p-adic analytic extension appeared in a formula of Casson Nogués [7] for the p-adic L-functions at the point 0.

Before Barnes, these functions had been introduced under a different form by Alexeiewsky [1], Glaisher [14], Hölder [17] and Kinkelin [18].

Barnes [4] defines the double Gamma function $\Gamma_2 = 1/G$ satisfying each of the following properties:

(a) $G(z+1) = \Gamma(z)G(z)$, for all complex z,

(b) $G(1) = 1$,

(c) As $n \to \infty$,

$$\begin{aligned}\log G(z+n+2) =& \frac{n+1+z}{2}\log 2\pi \\ &+ \left[\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z\right]\log n \\ &- \frac{3z^2}{4} - n - nz - \log A + \frac{1}{12} + O(\frac{1}{n}),\end{aligned}$$

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where Γ is the well-known Gamma function whose Weierstrass' canonical product form is

$$\Gamma(z)=\frac{e^{-\gamma z}}{z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{-1}e^{\frac{z}{k}} \tag{1}$$

and γ is the Euler-Mascheroni's constant defined by

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\cong 0.577\,215\,664\ldots \tag{2}$$

and A is called Glaisher's (or Kinkelin's) constant defined by

$$\log A=\lim_{n\to\infty}\log(1^1 2^2\cdots n^n)-\left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\log n+\frac{n^2}{4}, \tag{3}$$

the numerical value of being $1.282\,427\,130\cdots$.

From this definition Barnes deduced the Weierstrass' canonical product form of the double Gamma function:

(4)

$$\Gamma_2(z+1)^{-1}=G(z+1)=(2\pi)^{\frac{z}{2}}e^{-\frac{1}{2}\{(1+\gamma)z^2+z\}}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)^{k}e^{-z+\frac{z^2}{2k}}.$$

The Legendre duplication formula for Γ is given [25, p. 240]:

$$2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=\sqrt{\pi}\,\Gamma(2z). \tag{5}$$

The object of the present note is to prove a duplication formula for Γ_2:

$$\Gamma_2(a)\Gamma_2\left(a+\frac{1}{2}\right)^2\Gamma_2(a+1)=e^{-\frac{1}{4}}A^3 2^{2a^2-3a+\frac{11}{12}}\pi^{\frac{1}{2}-a}\Gamma_2(2a). \tag{6}$$

We can define the Gamma function Γ by using the Bohr-Mollerup theorem ([3, p. 14] and [12, p. 179]). By analogy Vignéras ([23, p. 239], Proposition 2.8) gives the criteria for the double Gamma function and more generally for the n-ple Gamma functions Γ_n, $n\geq 1$. For our purpose we reduce her proposition to the double Gamma function as in the following:

THEOREM. *There exists a unique meromorphic function $f(z)$ such that*

(a) $f(1) = 1$,

(b) $f(z+1) = \Gamma(z) f(z)$, *for all complex z*,

(c) *For $x \geq 1$, $f(x)$ is infinitely differentiable,*

$$\frac{d^3 f(x)}{dx^3} \log f(x) \geq 0.$$

It is not difficult to check that the G-function in (4) satisfies all the criteria in Theorem and so $f(z) = G(z)$ for all complex z.

From the Hermite formula for $\zeta(s,a)$ [25, p. 271] we deduce

$$\left\{ \frac{d}{ds} \zeta(s,a) \right\}_{s=0} = \log \Gamma(a) - \frac{1}{2} \log(2\pi) \quad \text{or} \quad \Gamma(a) = \sqrt{2\pi} e^{\zeta'(0,a)}, \tag{7}$$

where $\zeta(s,a) = \sum_{k=0}^{\infty} (a+k)^{-s}$, $a > 0$ is the generalized (or Hurwitz) zeta function which is analytic for $\mathrm{Re}(s) > 1$. It should be remarked in passing [25, pp. 265-280] that $\zeta(s,a)$ can be continued analytically to the whole s-plane except a simple pole at $s = 1$ with its residue 1. $\zeta(s,1) = \sum_{k=1}^{\infty} k^{-s} = \zeta(s)$ is the Riemann zeta function.

The double Hurwitz zeta function $\zeta_2(s,a)$ is defined by

$$\zeta_2(s,a) = \sum_{k_1,k_2=0}^{\infty} (a + k_1 + k_2)^{-s} \tag{8}$$

which is analytic for $\mathrm{Re}(s) > 2$ by the Eisenstein's theorem [13, p. 99]. Furthermore $\zeta_2(s,a)$ can be continued analytically to the whole s-plane except simple poles at $s = 1, 2$ by the contour integral representation [9]:

$$\zeta_2(s,a) = \frac{i\Gamma(1-s)}{2\pi} \int_C \frac{(-z)^{s-1} e^{-az}}{(1-e^{-z})^2} dz, \tag{9}$$

where the contour C is the same as in [25, p. 245].

We can reduce $\zeta_2(s,a)$ to $\zeta(s,a)$ [10]:

$$\zeta_2(s,a) = \zeta(s-1,a) + (1-a)\zeta(s,a). \tag{10}$$

From ([24, p. 462], Eq. (A.11)) we have

$$\log A = \frac{1}{12} - \zeta'(-1). \tag{11}$$

Now we can obtain a relationship between $\Gamma_2(a)$ and $\zeta_2'(0,a)$ similar to the formula (7):

$$\Gamma_2(a) = e^{-\frac{1}{12}} A (2\pi)^{\frac{1}{2}-\frac{1}{2}a} e^{\zeta_2'(0,a)} \tag{12}$$

where $\zeta_2'(s,a) = \frac{\partial}{\partial s}\zeta_2(s,a)$ and A is the Kinkelin's constant in (3).

Indeed, let $f(a)$ be the right side of (12). It is not difficult to show that $f(a)$ satisfies the criteria of Theorem: It follows from (10) that $\zeta_2'(0,1) = \zeta'(-1)$. Considering (11) we have $e^{\zeta_2'(0,1)} = A^{-1} e^{\frac{1}{12}}$. Therefore we have $f(1) = 1$.

It can be easily verified that

$$\zeta(s,a) = \zeta(s, m+a) + \sum_{n=0}^{m-1} (a+n)^{-s}, \; m = 1, 2, \ldots \tag{13}$$

Letting $m = 1$ in this formula (13), we find that $\zeta(s,a) = \zeta(s, 1+a) + a^{-s}$. Then considering (10) we see that $f(a+1)^{-1} = \Gamma(a) f(a)^{-1}$.

We have for $a > 0$

$$\frac{d^3}{da^3} \log f(a)^{-1} = -\frac{d^3}{da^3} \frac{d}{ds} \zeta_2(s,a)\bigg|_{s=0} = \sum_{k_1,k_2=0}^{\infty} \frac{2}{(a+k_1+k_2)^3} > 0.$$

Also by the analytic continuation of $\zeta_2(s,a)$ we see that $f(a)^{-1} \in C^{\infty}(0,\infty)$. Therefore, by Theorem, this completes the proof of (12).

Finally we will show the duplication formula (6). Indeed, we observe that

$$\begin{aligned}
&\zeta_2(s,a) + 2\zeta_2\left(s, a+\frac{1}{2}\right) + \zeta_2(s, a+1) \\
&= 2^s \Bigg\{ \sum_{k_1,k_2=0}^{\infty} (2a + 2k_1 + 2k_2)^{-s} + 2 \sum_{k_1,k_2=0}^{\infty} (2a + 1 + 2k_1 + 2k_2)^{-s} \\
&\qquad + \sum_{k_1,k_2=0}^{\infty} (2a + 2 + 2k_1 + 2k_2)^{-s} \Bigg\} \\
&= 2^s \zeta_2(s, 2a).
\end{aligned}$$

Differentiating both sides of the formula just obtained with respect to s and letting $s = 0$ in the resulting equation, and taking exponentials on both sides of the last resulting equation, we obtain

$$\begin{aligned}(14)\qquad &\exp\left(\zeta_2'(0,a)\right)\left[\exp\zeta_2'\left(0, a+\frac{1}{2}\right)\right]^2 \exp\left(\zeta_2'(0,a+1)\right)\\ &= 2^{\zeta_2(0,2a)}\exp\left(\zeta_2'(0,2a)\right).\end{aligned}$$

Recall the formula (cf. [2, p. 264], Eq. (17)): For every integer $m \geq 0$,

$$(15)\qquad \zeta(-m,a) = -\frac{B_{m+1}(a)}{m+1},$$

where $B_{m+1}(a)$ are Bernoulli polynomials.

Now the desired duplication formula (6) follows immediately from formulas (12), (14) and (15).

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