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SOME EXAMPLES OF ANALYTIC IRREDUCIBLE PLANE CURVE SINGULARITIES

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1. Introduction

The aim in this paper is to classify analytically some special type of irreducible plane curve singularities in terms of Weierstrass polynomials. Let $V = \{f(z,y) = 0\}$ be an analytically irreducible plane curve with isolated singularity at the origin and C be the curve defined by $y = t^n, z = t^\alpha + t^\beta$ using the Puiseux expansion where $n < \alpha < \beta$ and $n > (n,\alpha) > (n,\alpha,\beta) = 1$. Assume that $f = (z^a + y^b)^c + y^d z^e$ where $1 < a < b, (a,b) = 1, ad + be - abc > 0$ and $(ad + be, c) = 1$. Then V and C have the topologically equivalent singularity at the origin if and only if $(n,\alpha) = c, n = ac, \alpha = bc$ and $\beta = bc + ad + be - abc$.

Let $V = \{f(z,y) = 0\}$ and $W = \{g(z,y) = 0\}$ be analytically irreducible curves with isolated singularity at the origin where f and g are square free in ${}_2\mathcal{O}$, the ring of germs of holomorphic functions at the origin. If V and W are topologically equivalent at the origin, then denote this relation by $f \sim g$ for brevity. If V and W are analytically equivalent at the origin, then we write $f \approx g$. Otherwise, we write $f \not\approx g$. Observe [4] that if $f \sim (z^a + y^b)^c + y^d z^e$ where $1 < a < b, (a,b) = 1$, $ad + be > abc$ and $(ad + be, c) = 1$, then by a nonsingular linear change of coordinate f can be represented uniquely as

$$u\{(z^a + y^b)^c + A_1(z^a + y^b)^{c-1} + \cdots + A_{c-1}(z^a + y^b) + A_c\}$$

where u is a unit in ${}_2\mathcal{O}$ and each $A_i \in {}_1\mathcal{O}[z]$ is a polynomial in z of degree $< a$ with coefficient holomorphic in ${}_1\mathcal{O}$, the ring of germs of

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holomorphic functions of y at $y = 0$ by the Weierstrass preparation and division theorem. Now, we are going to prove the following :

(1) If $f = (z^a + y^b)^c + uy^d z^e$ where $ad + be - abc > 0$ and u is a nonzero number then $f \approx (z^a + y^b)^c + y^d z^e$.

(2) Let $f = (z^a + y^b)^c + y^d z^e$ and $g = (z^a + y^b)^c + y^{d_1} z^{e_1}$ where $1 < a < b, (a, b) = 1, ad + be - abc > 0$ and $(ad + be, c) = 1$. Then

  (i) If $e < ac - a$ and $e_1 < ac - a$, then $f \approx g$ if and only if $d = d_1$ and $e = e_1$.

  (ii) If $e_1 \geq ac - a > e$, then $f \sim g$ implies that either $f \approx g$ or $f \not\approx g$.

  (iii) If $ac > e_1 \geq ac - a > e$ and $d_1 + e_1 < ac - a + b$, then $f \sim g$ does not imply that $f \approx g$.

(3) Let $f = (z^a + y^b)^c + A_2 y^{\alpha_2} z^{\beta_2} (z^a + y^b)^{c-2} + \cdots + A_{c-1} y^{\alpha_{c-1}} z^{\beta_{c-1}} (z^a + y^b) + A_c y^d z^e$ and $g = (z^a + y^b)^c + B_2 y^{\gamma_2} z^{\delta_2} (z^a + y^b)^{c-2} + \cdots + B_{c-1} y^{\gamma_{c-1}} z^{\delta_{c-1}} (z^a + y^b) + B_c y^{d_1} z^{e_1}$ where A_i and B_j are units in ${}_2\mathcal{O}$ if exist for $2 \leq i < c$, $2 \leq j < c$, and A_c and B_c are units in ${}_2\mathcal{O}$. Suppose that f and g satisfy conditions (i), (ii), (iii) in Theorem 3.8. Let $I = \{i : A_i \text{ is a unit}\}$ and $J = \{j : B_j \text{ is a unit}\}$. Then

  (i) If $f \approx g$ and $ad + be - abc > c$, then $I = J$ and for each $i \in I$, $a\alpha_i + b\beta_i = a\gamma_i + b\delta_i$.

  (ii) If $f \approx g$ and $ad + be - abc > c$, then $I = J$, and $(\alpha_i, \beta_i) = (\gamma_i, \delta_i)$ and $(d, e) = (d_1, e_1)$ as sets whenever β_i, δ_j, e and e_1 are smaller than a for all i and j.

2. Known preliminaries

DEFINITION 2.1. Let $V = \{z \in \mathbb{C}^n : f(z) = 0\}$ and $W = \{z \in \mathbb{C}^n : g(z) = 0\}$ be germs of complex analytic hypersurfaces with isolated singular point at the origin.

(i) V and W are said to be topologically equivalent at the origin if there is a germ at the origin of homeomorphisms $\phi : (U_1, 0) \to (U_2, 0)$ such that $\phi(V) = W$ where U_1 and U_2 are open subsets in $\mathbb{C}^n$. In this case denote this relation by $f \sim g$.

(ii) V and W are said to be analytically equivalent at the origin if there is a germ at the origin of biholomophisms $\phi : (U_1, 0) \to (U_2, 0)$ such that $\phi(V) = W$ where U_1 and U_2 are open subsets in $\mathbb{C}^n$. Then denote this relation by $f \approx g$. If not, we write $f \not\approx g$. Let ${}_n\mathcal{O}$ denote the ring of germs of holomorphic functions at the origin in $\mathbb{C}^n$.

THEOREM 2.2 (ZARISKI). *Let* $f(y,z) = z^n + a_2 y^{\alpha_2} z^{n-2} + \cdots + a_n y^{\alpha_n}$ *be irreducible in* ${}_2\mathcal{O}$ *where each* $a_i = a_i(y)$ *is a unit in* ${}_1\mathcal{O}$ *if exists and the* α_i *are positive integers. Assume that the multiplicity of* f *at the origin is* $n \geq 2$. *Then the curve defined by* f *at the origin can be described topologically by* $y = t^n$ *and* $z = t^{\beta_1} + t^{\beta_2} + \cdots + t^{\beta_s}$ *with* $\beta_1 = \alpha_n$ *where* $n < \beta_1 < \beta_2 < \cdots < \beta_s$ *and* $n > (n, \beta_1) > \cdots > (n, \beta_1, \cdots, \beta_s) = 1$. *If for a given* f *there is another homeomorphic paramerization defined by* $y = t^m$ *and* $z = t^{\gamma_1} + t^{\gamma_2} + \cdots + t^{\gamma_b}$ *where* $m < \gamma_1 < \gamma_2 < \cdots < \gamma_b$ *and* $m > (m, \gamma_1) > \cdots > (m, \gamma_1, \cdots, \gamma_b) = 1$, *then* $n = m, s = b$ *and* $\beta_i = \gamma_i$. *Conversely, the curve defined by the parameter with the same inequality as above must be irreducible at the origin.*

THEOREM 2.3 (ARNOLD). *Assume that* $1 < n < k, (n,k) = 1$ *and that* $g \sim z^n + y^k$ *where* g *is holomorphic at the origin in* $\mathbb{C}^2$. *Then* $g \approx z^n + y^k + \Sigma c_i y^{\alpha_i} z^{\beta_i}$ *where each* c_i *is a nonzero number if exists with* $1 \leq \beta_i \leq n-2$, $1 \leq \alpha_i \leq k-2$ *and* $n\alpha_i + k\beta_i > nk$.

THEOREM 2.4 (MATHER-YAU). *Suppose that* $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *are holomorphic functions with isolated singular point at the origin. Then the following conditions are equivalent.*

(i) $f \approx g$.
(ii) ${}_n\mathcal{O}/(f, \triangle f)$ *is isomorphic to* ${}_n\mathcal{O}/(g, \triangle g)$ as a $\mathbb{C}$-algebra *where* $(f, \triangle f)$ *is the ideal generated by* $f, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}$. *Furthermore, if* $f \circ \phi = ug$ *for some unit* u *and a biholomorphic map* ϕ, *then* $\phi^*(f, \triangle f) = (g, \triangle g)$ *where* $\phi^*(h) = h \circ \phi$.
(iii) ${}_n\mathcal{O}/(f, m\triangle f)$ *is isomorphic to* ${}_n\mathcal{O}/(g, m\triangle g)$ as a $\mathbb{C}$-algebra *where* $(f, m\triangle f)$ *is the ideal generated by* f *and* $z_i \frac{\partial f}{\partial z_j}$ *for all* $i, j = 1, \cdots, n$.

THEOREM 2.5 [4]. *Let* $f(z,y) = z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_n y^{\alpha_n}$ *be irreducible in* ${}_2\mathcal{O}$ *where each* a_i *is a unit in* ${}_2\mathcal{O}$ *if exists and the* α_i *are* *positive integers. Then* $\frac{\alpha_i}{i} \geq \frac{\alpha_n}{n}$ *for all* i. *Moreover, if* $\alpha_n = nk$ *for some integer* k, *then* $\frac{\alpha_n}{n} = \frac{\alpha_i}{i}$ *for all* $i = 1, \cdots, n-1$.

COROLLARY 2.6. *Let* $f(z,y) = z^n + a_1 y^{\alpha_1} z^{n-1} + \cdots + a_{n-1} y^{\alpha_{n-1}} z + y^k$ *with* $(n,k) = 1$ *where each* a_i *is a unit in* ${}_2\mathcal{O}$ *if exists and the* α_i *are positive integers. Then* f *is irreducible in* ${}_2\mathcal{O}$ *if and only if* $\frac{k}{n} < \frac{\alpha_i}{i}$ *for all* $i \neq n$. *Moreover, in this case* $f \sim z^n + y^k$ *in* ${}_2\mathcal{O}$.

THEOREM 2.7 [5]. *Let* $f = z^n + y^k + uy^\alpha z^\beta$ *and* $g = z^n + y^k + vy^\gamma z^\delta$ *where* $n < k, (n,k) = 1$, *and* u, v *are units in* ${}_2\mathcal{O}$, *and* $1 \leq \beta, \delta \leq n-2$ *and* $1 \leq \alpha, \gamma \leq k-2$ *with* $n\alpha + k\beta > nk$ *and* $n\gamma + k\delta > nk$. *Then* $f \approx g$ *if and only if* $\alpha = \gamma$ *and* $\beta = \delta$.

THEOREM 2.8 [6]. *Let* $f = z^n + A_1 y^{\alpha_1} z^{p_1} + \cdots + A_t y^{\alpha_t} z^{p_t} + y^k$ *where* $n < k, (n,k) = 1, t \geq 2, n-2 \geq p_1 > p_2 > \cdots > p_t \geq 1$ *and* $\alpha_1 + p_1 < \alpha_2 + p_2 < \cdots < \alpha_t + p_t < k$ *with* $\frac{\alpha_1}{n-p_1} > \frac{\alpha_2}{n-p_2} > \cdots > \frac{\alpha_t}{n-p_t} > \frac{k}{n}$ *and each* $A_i = A_i(z,y)$ *is a unit in* ${}_2\mathcal{O}$ *for* $i = 1, \cdots, t$. *Let* $g = z^n + B_1 y^{\beta_1} z^{q_1} + \cdots + B_s y^{\beta_s} z^{q_s} + y^k$ *where* $s \geq 2, n-2 \geq q_1 > q_2 > \cdots > q_s \geq 1$ *and* $\beta_1 + q_1 < \beta_2 + q_2 < \cdots < \beta_s + q_s < k$ *with* $\frac{\beta_1}{n-q_1} > \frac{\beta_2}{n-q_2} > \cdots > \frac{\beta_s}{n-q_s} > \frac{k}{n}$ *and each* $B_j = B_j(z,y)$ *is a unit in* ${}_2\mathcal{O}$ *for* $j = 1, \cdots, s$. *If* $f \approx g$, *then* $t = s$, *and for each* $i = 1, \cdots, t, (\alpha_i, p_i) = (\beta_i, q_i)$ *and* $A_i(0,0)^{n\alpha_t + kp_t - nk} B_t(0,0)^{n\alpha_i + kpi - nk} = B_i(0,0)^{n\alpha_t + kp_t - nk} A_t(0,0)^{n\alpha_i + kp_i - nk}$.

3. Analytic classification of some irreducible plane curve singularities topologically equivalent to $y = t^n, z = t^\alpha + t^\beta$ with $n < \alpha < \beta$ and $n > (n,\alpha) > (n,\alpha,\beta) = 1$

PROPOSITION 3.1. *Let* $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ *be a germ of holomorphic function with isolated singular point at the origin. Assume that* $f \sim (z^a + y^b)^c + y^d z^e$ *where* $1 < a < b, (a,b) = 1, 1 < c$ *and* $(r,c) = 1$ *with* $r = ad + be - abc > 0$. *Then* $\{f = 0\} \sim \{y = t^{ac}, z = t^{bc} + t^{bc+r}\}$ *at the origin. Conversely, if* $\{f = 0\} \sim \{y = t^n, z = t^\alpha + t^\beta\}$ *where* $n < \alpha < \beta$ *and* $n > (n,\alpha) > (n,\alpha,\beta) = 1$, *then* $f \sim (z^a + y^b)^c + y^d z^e$ *where* $n = ac, \alpha = bc$ *with* $c = (n,\alpha)$, *and* d *and* e *are positive integers such that* $ad + be = abc + \beta - \alpha$.

Proof. See Theorem 2.2 and [4].

REMARK 3.2. Let $f = (z^a + y^b)^c + y^d z^e$ where $1 < a < b, (a, b) = 1, 1 < c$ and $(r, c) = 1$ with $r = ad + be - abc > 0$.

(i) If $(z^{a_1} + y^{b_1})^{c_1} + y^{d_1} z^{e_1} \sim f$ with $a_1 < b_1$, then $a_1 = a, b_1 = b, c_1 = c$ and $ad_1 + be_1 = ad + be$. Furthermore, the condition that $ad + be = ad_1 + be_1$ implies that $d_1 = d + kb$ and $e = e_1 + ka$ for some integer k. For example, if $e_1 > e$, then $d_1 + e_1 < d + e$.

(ii) If f is a Weierstrass polynomial in z of multiplicity ac at the origin, then we may assume that $0 \leq e < ac$.

DEFINITION 3.3. Let $f = z^n + A_1 y^{\alpha_1} z^{n-1} + \cdots + A_{n-1} y^{\alpha_{n-1}} z + A_n y^{\alpha_n}$ and $g = z^n + B_1 y^{\beta_1} z^{n-1} + \cdots + B_{n-1} y^{\beta_{n-1}} z + B_n y^{\beta_n}$ be irreducible in ${}_2\mathcal{O}$ where the $A_i = A_i(z, y)$ and $B_i = B_i(z, y)$ are units in ${}_2\mathcal{O}$ if exist. If $f \approx g$, then by definition, there is a biholomorphic mapping $\phi : (U_1, 0) \to (U_2, 0)$ such that $f \circ \phi = ug$ where U_1 and U_2 are open subsets in $\mathbb{C}^2$ and u is a unit in ${}_2\mathcal{O}$. Write $\phi(z, y) = (H, L)$ as follows:

$$H = H(z, y) = \alpha z + \beta y + H_2 + H_3 + \cdots \quad \text{and}$$
$$L = L(z, y) = \gamma z + \delta y + L_2 + L_3 + \cdots$$

where H_n and L_n are homogeneous polynomials of degree n with $H_n = \alpha_{n,0} z^n + \alpha_{n-1,1} z^{n-1} y + \cdots + \alpha_{0,n} y^n$ and $L_n = \gamma_{n,0} z^n + \gamma_{n-1,1} z^{n-1} y + \cdots + \gamma_{0,n} y^n$. Note that $\alpha\delta - \beta\gamma \neq 0$. Then $f \circ \phi(z, y) = H^n + A_1 L^{\alpha_1} H^{n-1} + \cdots + A_n L^{\alpha_n} = u(z^n + B_1 y^{\beta_1} z^{n-1} + \cdots + B_n y^{\beta_n})$ where the $A_i = A_i(H, L)$ are units in ${}_2\mathcal{O}$ if exist. It is trivial that if $n < \alpha_i + n - i$ for all i, $n < \alpha_n, n < \beta_i + n - i$ for all i and $n < \beta_n$, then $\beta = 0$.

DEFINITION 3.4. If the coefficent of monomial $y^l z^m$ is zero in the convergent power series expansion of $f(z, y) \in {}_2\mathcal{O}$, then write $y^l z^m \notin f(z, y)$. Otherwise, we write $y^l z^m \in f(z, y)$.

PROPOSITION 3.5. *Let $f = (z^a + y^b)^c + u y^d z^e$ for any nonzero number u, and $ad + be - abc > 0$. Then $f \approx (z^a + y^b)^c + y^d z^e$.*

Proof. Let $\phi(z, y) = (\varepsilon^b z, \varepsilon^a y)$ for some nonzero number ε. Then ϕ is a biholomorphic map. We get

$$f \circ \phi = \varepsilon^{abc}\{(z^a + y^b)^c + u\varepsilon^{ad+be-abc} y^d z^e\}.$$

Choose ε such that $u\varepsilon^{ad+be-abc} = 1$. So, it is done.

Before proving the next theorem, we need the following.

LEMMA 3.6. *Let* $f = (z^a + y^b)^c + y^d z^e$ *with* $ad + be - abc > 0$. *Then* $(f, f_z, f_y) = (y^d z^e, f_z, f_y)$ *where* (f, f_z, f_y) *is the ideal in* ${}_2\mathcal{O}$.

Proof. Then $zf_z = acz^a(z^a + y^b)^{c-1} + ey^d z^e$ and $yf_y = bcy^b(z^a + y^b)^{c-1} + dy^d z^e$. So $f - (\frac{1}{ac} z f_z + \frac{1}{bc} y f_y) = (1 - \frac{e}{ac} - \frac{d}{bc}) y^d z^e = -\frac{ad+be-abc}{abc} y^d z^e$. Since $ad + be - abc > 0, y^d z^e \in (f, f_z, f_y)$ and $f \in (y^d z^e, f_z, f_y)$. Thus it is done.

THEOREM 3.7. *Let* $f = (z^a + y^b)^c + y^d z^e$ *and* $g = (z^a + y^b)^c + y^{d_1} z^{e_1}$ *such that* $1 < a < b, 1 < c, (a, b) = 1, (ad + be, c) = 1$ *and* $(ad_1 + be_1, c) = 1$ *with* $ad + be - abc > 0$ *and* $ad_1 + be_1 - abc > 0$.

(i) *If* $e < ac - a$ *and* $e_1 < ac - a$, *then* $f \approx g$ *if and only if* $d = d_1$ *and* $e = e_1$.
(ii) *If* $e < ac - a$ *and* $e_1 \geq ac - a$, *then* $f \sim g$ *implies that either* $f \approx g$ *or* $f \not\approx g$.
(iii) *If* $ac > e_1 \geq ac - a > e$ *and* $d_1 + e_1 < ac - a + b$, *then* $f \sim g$ *does not imply that* $f \approx g$.

Proof. (i) It is enough to show that $f \approx g$ implies that $d = d_1$ and $e = e_1$. Suppose that $f \approx g$. Let $\phi(z, y) = (H, L)$ be a biholomorphic map such that $f \circ \phi = ug$ for some unit $u \in {}_2\mathcal{O}$ as we have seen in Definiton 3.3. Recall that $H = \alpha z + \beta y + H_2 + \cdots$ and $L = \gamma z + \delta y + L_2 + \cdots$. Note that $\beta = 0$, and $\delta \neq 0$ since $\alpha\delta - \beta\gamma \neq 0$. By Theorem 2.4 and Lemma 3.6,

$$\phi^*(y^d z^e) = L^d H^e = Ay^{d_1} z^{e_1} + Bg_z + Cg_y$$

where A,B and C are in ${}_2\mathcal{O}$. Observe that $y^d z^e \in L^d H^e$, and that $d = d_1$ if and only if $e = e_1$ by Remark 3.2. Suppose that $e \neq e_1$. Then we may assume that $d + e < d_1 + e_1$ for the proof because $d + e = d_1 + e_1 + k(b - a)$ for some integer k by Remark 3.2. So $y^d z^e$ would belong to $Bg_z + Cg_y$. Note that

$$Bg_z + Cg_y = (acBz^{a-1} + bcCy^{b-1})(z^a + y^b)^{c-1} + (e_1 By + d_1 Cz) y^{d_1 - 1} z^{e_1 - 1}.$$

But $y^d z^e \notin (e_1 By + d_1 Cz) y^{d_1 - 1} z^{e_1 - 1}$ because $d = d_1 + kb$ for some negative integer k by Remark 3.2. Thus $y^d z^e \in (acBz^{a-1} + bcCy^{b-1})(z^a +$

$y^b)^{c-1}$. Let X_m be the nonzero homogeneous polynomial of the smallest degree m in $acBz^{a-1} + bcCy^{b-1}$. If $m + ac - a > d + e$, it is trivial that $y^d z^e \notin Bg_z + Cg_y$. If $m + ac - a \leq d + e$, then $y^d z^e \in Bg_z + Cg_y$ and so $y^d z^e \in X_m z^{ac-a}$. But $e < ac - a$ and then it would give a contradiction. Therefore, $d = d_1$ and $e = e_1$.

(ii) Suppose that $e < ac - a \leq e_1$. Then it is enough to show that there are two examples such that the first one implies $f \not\approx g$ and the second one implies $f \approx g$.

(ii$_a$) Let $f(z,y) = (z^4 + y^7)^3 + y^{11} z^6$ and $g(z,y) = (z^4 + y^7)^3 + y^4 z^{10}$. Claim that $f \not\approx g$, but $f \sim g$. Clearly, $f \sim g$. Suppose that $f \approx g$. Then $f \circ \phi = ug$ for some biholomorphic mapping ϕ and a unit $u \in {}_2\mathcal{O}$ as in Definition 3.3. So $f(H,L) = (H^4 + L^7)^3 + L^{11}H^6 = H^{12} + 3L^7H^8 + L^{11}H^6 + 3L^{14}H^4 + L^{21} = ug(z,y) = u(z^{12} + y^4 z^{10} + 3y^7 z^8 + 3y^{14} z^4 + y^{21})$. Then $y^4 z^{10} \in ug$ and so $y^4 z^{10} \in H^{12}$. Recall that $H = \alpha z + \beta y + H_2 + H_3 + \cdots$ in Definition 3.3. Note that $\beta = 0$ and H_2, H_3 are identically zero. So $y^4 z^{10} \notin H^{12} = (\alpha z + H_4 + H_5 + \cdots)^{12}$. It is a contradiction.

(ii$_b$) Let $f(z,y) = (z^2 + y^3)^2 + \frac{1}{3} y^3 z^3$ and $g(z,y) = (z^2 + y^3)^2 - y^6 z$. Claim that $(f, m\Delta f) = (g, m\Delta g)$ where $(f, m\Delta f)$ is the ideal generated by f, zf_z, yf_y, yf_z and zf_y. By an elementary computation, it is easily shown that $(f, m\Delta f) = (z^4 + y^3 z^2, y^3 z^2 + y^6, y^3 z^3, y^6 z, 4yz^3 + 4y^4 z + y^4 z^2, y^2 z^3 + y^5 z, y^4 z^2 + y^7) = (g, m\Delta g)$. By Theorem 2.4, $f \approx g$.

(iii) To get a contradiction, suppose that $f \approx g$. Let $f \circ \phi = ug$ as in Definition 3.3. Then $(H^a + L^b)^c + L^d H^e = u((z^a + y^b)^c + y^{d_1} z^{e_1})$. Note that $y^{d_1} z^{e_1} \in ug$ because $e_1 < ac$ and $d_1 + e_1 < ac + b - a$. Since $e < e_1$ implies that $d_1 + e_1 < d + e$ by Remark 3.2, $y^{d_1} z^{e_1} \in H^{ac}$. Recall that $H = \alpha z + \beta y + H_2 + H_3 + \cdots$ from Definition 3.3. Then $\beta = 0$ and $H_2, H_3, \cdots, H_{b-a}$ are identically zero. So $y^{d_1} z^{e_1} \in (\alpha z + H_{b-a+1} + H_{b-a+2} + \cdots)^{ac}$. But $ac - 1 + b - a + 1 > d_1 + e_1$ implies that $y^{d_1} z^{e_1}$ cannot belong to H^{ac}. It is impossible.

THEOREM 3.8. *Let* $f(z,y) = (z^a + y^b)^c + A_2 y^{\alpha_2} z^{\beta_2} (z^a + y^b)^{c-2} + \cdots + A_{c-1} y^{\alpha_{c-1}} z^{\beta_{c-1}} (z^a + y^b) + A_c y^d z^e$ *where for* $2 \leq i \leq c-1$, *each* A_i *is a unit in* ${}_2\mathcal{O}$ *if exists, and* A_c *is a unit in* ${}_2\mathcal{O}$. *Let* $g(z,y) = (z^a + y^c)^c + b_2 y^{\gamma_2} z^{\delta_2} (z^a + y^b)^{c-2} + \cdots + B_{c-1} y^{\gamma_{c-1}} z^{\delta_{c-1}} (z^a + y^b) + B_c y^{d_1} z^{e_1}$

where for $2 \le j \le c-1$, each B_j is a unit in ${}_2\mathcal{O}$ if exists, and B_c is a unit in ${}_2\mathcal{O}$. Assume that the following conditions are satisfied :

(i) $1 < a < b, (a,b) = 1, c > 1, (ad+be, c) = 1, ad+be-abc > 0$ and $ad+be = ad_1 + be_1$.

(ii) $\frac{a\alpha_2+b\beta_2}{2} > \frac{a\alpha_3+b\beta_3}{3} > \cdots > \frac{a\alpha_{c-1}+b\beta_{c-1}}{c-1} > \frac{ad+be}{c}$ and $\frac{a\gamma_2+b\delta_2}{2} > \frac{a\gamma_3+b\delta_3}{3} > \cdots > \frac{a\gamma_{c-1}+b\delta_{c-1}}{c-1} > \frac{ad+be}{c}$.

(iii) $a\alpha_2 + b\beta_2 - 2ab + c - 2 < \cdots < a\alpha_{c-1} + b\beta_{c-1} - (c-1)ab + 1 < ad+be-abc$ and $a\gamma_2 + b\delta_2 - 2ab + c - 2 < \cdots < a\gamma_{c-1} + b\delta_{c-1} - (c-1)ab + 1 < ad_1 + be_1 - abc$.

Let $I = \{i : A_i$ is a unit, $2 \le i \le c-1\}$ and $J = \{j : B_j$ is a unit, $2 \le j \le c-1\}$. Then, we have the following:

(a) $f \sim g \sim (z^a + y^b)^c + y^d z^e$, *which is irreducible in* ${}_2\mathcal{O}$.

(b) *If $f \approx g$ an $ad+be-abc > c$, then $I = J$ and for each $i \in I, a\alpha_i + b\beta_i = a\gamma_i + b\delta_i$.*

Proof. Note that blow-up process preserves analytic equivalence. Because for $1 \le i \le c-2$, $\frac{a\alpha_i+b\beta_i}{i} > \frac{ad+be}{c}$ and $ad+be-abc > 0$ imply that $a\alpha_i + b\beta_i > iab$, it is possible to use the σ-process for $V = \{f = 0\}$ and $W = \{g = 0\}$ until the proper transform $V_1 = \{f_1 = 0\}$ of $V$and the proper transform $W_1 = \{g_1 = 0\}$ of W are both toplogically equivalent to $\{p^c + q^{ad+be-abc} = 0\}$ at $(p,q) = (0,0)$, using the pq-coordinates. In detail, these proper transforms V_1 and W_1 can be represented analytically after the same number of blow-ups, using the same coordinates as follows :

$$f_1 : p^c + D_2 q^{a\alpha_2+b\beta_2-2ab} p^{c-2} + \cdots + D_{c-1} q^{a\alpha_{c-1}+b\beta_{c-1}-(c-1)ab} p + D_c q^{ad+be-abc}$$

$$g_1 : p^c + E_2 q^{a\gamma_2+b\delta_2-2ab} p^{c-2} + \cdots + E_{c-1} q^{a\gamma_{c-1}+b\delta_{c-1}-(c-1)ab} p + E_c q^{ad_1+be_1-abc}$$

where D_i and E_j are units if exist for $2 \le i \le c-1$ and $2 \le j \le c-1$, and D_c and E_c are units in $\mathbb{C}\{p,q\}$, the ring of germs of holomorphic functions at $(p,q) = (0,0)$, by (i) and (ii). Then $f_1 \sim g_1 \sim p^c + q^{ad+be-abc}$ by (ii) and Corollary 2.6 because $(ad+be, c) = (ad_1+be_1, c) =$

1. So $f \sim g \sim (z^a + y^b)^c + y^d z^e$, which is irreducible in ${}_2\mathcal{O}$. Thus we proved (a). Next, to prove (b), if $f \approx g$ and $ad + be - abc > c$, then note that f_1 and g_1 satisfy the assumptions of Theorem 2.8 or 2.7. Therefore, $I = J$, and $a\alpha_i + b\beta_i = a\gamma_i + b\delta_i$ for each $i \in I$.

Observe that if f and g in Theorem 3.8 are Weierstrass polynomials in z of degree ac, then by the Weierstrass division theorem it is interesting to consider the following case.

COROLLARY 3.9. *Assume that the conditions* (i), (ii), (iii) *as in Theorem 3.8 are satisfied for* $f(z, y)$ *and* $g(z, y)$. *Let* $I = \{i : A_i \text{ is a unit}, 2 \leq i \leq c-1\}$ *and* $J = \{j : B_j \text{ is a unit}, 2 \leq j \leq c-1\}$. *Suppose that for each* $i \in I$ *and* $j \in J, \beta_i < a, \delta_j < a$ *and that* $e < a, e_1 < a$. *If* $f \approx g$ *and* $ad + be - abc > c$, *then* $I = J$ *and for each* $i \in I, (\alpha_i, \beta_i) = (\gamma_i, \delta_i)$ *and* $(d, e) = (d_1, e_1)$ *as sets.*

Proof. By Theorem 3.8, $I = J$ and for each $i \in I, a\alpha_i + b\beta_i = a\gamma_i + b\delta_i$ and $ad + be = ad_1 + be_1$. So $a(\alpha_i - \gamma_i) = b(\delta_i - \beta_i)$. Since $(a, b) = 1, 0 \leq \beta_i < a$ and $0 \leq \delta_i < a$, then $\alpha_i = \gamma_i$ and $\beta_i = \delta_i$ for each $i \in I$. Similarly, we can prove that $d = d_1$ and $e = e_1$.

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