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THE OPTION VALUATION WHEN THE SECURITY MODEL IS A PROCESS OF MIXED TYPE

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1. Introduction

The history of option valuation problem goes back to the year 1900 when Louis Bachelier deduced an option valuation formula under the assumption that the price process follows standard Brownian motion. More than 50 years later, the research for a mathematical theory of option valuation was taken up by Samuelson ([6]) and others. This work wsa brought into focus in the major paper by Black and Scholes ([1]) in which a complete option valuation model was derived on the assumption that the underlying price model is a geometric Brownian motion. This paper starts with subjects developed mainly in Harrison and Kreps ([4]) and in Harrison and Pliska ([5]). The ideas established in these papers are essential for option valuation problem, and in particularfor the point of view that we take in this paper.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $\{\mathcal{F}_t : 0 \le t \le T\}$ satisfying the usual conditions:

$$\mathcal{F}_0 \text{ contains all the null sets of } P$$
$$\{\mathcal{F}_t\} \text{ is right-continuous}$$

Let $S = \{S_t : 0 \le t \le T\}$ be a vector price process whose components $S^0, S^1, \cdots, S^K$ are adapted, right continuous with left limits and strictly positive. We assume that S^0 has finite variation and is continuous. As a convenient normalization, let $S^0(0) = 1$ throughout. If S^0 was absolutely continuous, then we could write

$$S^0(t) = \exp\left(\int_0^t r(s)ds\right), \, 0 \le t \le T$$

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for some process r, then r_t would be interpreted as the riskless interest rate at time t. Defining $\beta_t = 1/S^0(t)$, we call β the *discount process* for S. Then discounted price process $S^*(t)$ for this security is

$$S^*(t) = \beta(t)S(t)$$

Let P^* be a probability measure on $(\Omega, \mathcal{F})$ which is equivalent to P and such that $Z(t)$ is a martingale under P^*, denoting by $E^*(\cdot)$ the associated expectation operator with respect to P^*. We define $\mathcal{L}(Z)$ as the set of all predictable processes $G = (G^1, \cdots, G^K)$ such that the increasing process

$$\left(\int_0^t (G_s^k)^2 d[S^{*k}, S^{*k}]_s\right)^{\frac{1}{2}}, \ 0 \le t \le T$$

is locally integrable under P^* where $[\cdot, \cdot]_s$ is quadratic covariational process. A *trading strategy* is defined as a $K+1$ dimensional process $\phi = \{\phi_t : 0 \le t \le T\}$ such that $(\phi^1, \cdots, \phi^K) \in \mathcal{L}(Z)$. Defining $V^*(\phi) = \beta\phi S$ and $G^*(\phi) = \int_0^t \phi dZ$, a trading strategy ϕ is said to be *admissible* if $V^*(\phi) \ge 0$, $V^*(\phi) = V_0^*(\phi) + G^*(\phi)$, and $V^*(\phi)$ is a martingale under P^*. A *contingent claim* is defined as a positive random variable X. Let Φ^* be the class of all admissible trading strategies. A claim is said to be *attainable* if there exists $\phi \in \Phi^*$ such that $V_T^*(\phi) = \beta_T X$, in which case ϕ is said to *generate* X and $\pi = V_0^*(\phi)$ is called the *price associated with* X. Since possession of portfolio that contains only stock and bond is completely equavalent to possession of the call option, the market value of its constituent securities at time zero is the unique rational value for the option.

In this note, we derive option valuation formula when the security model is a geometric Brownian motion and conditional Poisson process. In particular, we apply this formula to European call option.

2. The Main Results

We begin with:

LEMMA 1. *Let $X = \{X_t : 0 \le t \le T\}$ be a semimartingale and consider the equation*

$$U_t = U_0 + \int_0^t U_s dX_s, \ 0 \le t \le T, \tag{1}$$

where $U_0 \in \mathcal{F}_0$ is given. Then equation (1) has only semimartingale solution U and it is given by

$$U_t = U_0 \mathcal{E}(X),\ 0 \le t \le T,$$

where

$$\mathcal{E}(X) = \exp(X_t - X_0 - \frac{1}{2}[X,X]_t) \prod_{s \le t} (1 + \Delta X_s) \exp(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2)$$

and

$$\Delta X_s = X_s - X_{s-}.$$

Proof. See Harrison and Pliska ([5]).

The price process we consider is the following:

$$\frac{dS}{S(t)} = \mu(t,\omega)dt + \sigma(t,\omega)dB(t) + \int_R \gamma(t;y)\nu(dy;dt) \tag{2}$$

where μ, σ and γ are predictable processes and $B(t)$ is a $(P, \mathcal{F}_t)$-standard Brownian motion. The last term in (2) gives rise to jumps in S of random relative size γ at random time points τ_n. $\nu(A;t)$ equals the number of jumps of a marked conditional Poisson process N_t with conditional intensity kernal λ_t makes with values in the Borel set $A \subset R$ before time t, R is the mark-space.
Denote the Radon-Nikodym derivative of P^* with respect to P by L_T and define, for each $t \in [0,T]$,

$$L_t = L_0 \exp\Big(-\int_0^t \Big(\frac{\mu - r}{\sigma}\Big) dB - \frac{1}{2}\int_0^t \Big(\frac{\mu - r}{\sigma}\Big)^2 ds\Big) \prod_{n=1}^{N_t} h(\tau_n, y_n)$$

where L_0 is $\mathcal{F}_0$-measurable, $EL_0 = 1$, and the nonnegative predictable process $h(t,y)$ satisfies

$$\int_R h(t,y)H_t(dy) = 1,\ \int_R \gamma(t,y)h(t,y)H_t(dy) = 0$$

where $H_t(dy)$ is a probability transition kernel, that is, conditional distribution of the embedded jump size given that t is a time of jump and hence $(\lambda_t, H_t(dy))$ is a local characteristic of $\nu(dy : dt)$.

We now meet:

THEOREM 2. *Assume that*

$$\int_0^T \left(\frac{\mu - r}{\sigma}\right)^2 ds < \infty.$$

If $E(L_T) = 1$, then the price π_T^{ψ} at time zero of a contingent claim $X = \psi(S_T)$ which is the function of security price S_T expiring at time t is given as

$$\begin{aligned}
\pi_T^{\psi} = & e^{-\int_0^T r ds} \sum_{k=0}^{\infty} e^{-\int_0^t \lambda_s ds} \frac{(\int_0^t \lambda_s ds)^k}{k!} \int \cdots \int_R \\
& \left(\int_y \psi(S(0) \prod_{n=1}^{k} (1 + \gamma(\tau_n, y_n) e^{y - \frac{1}{2}\rho_T^2} \cdot S^0(T)) \frac{e^{-y^2/2\rho_T^2}}{\sqrt{2\pi}\rho_T} dy \right) \qquad (3) \\
& H_t^*(dy_1) \times \cdots \times H_t^*(dy_k)
\end{aligned}$$

where $H_t^*(dy) = h(t,y)H_t(dy)$ and $\rho_T^2 = \int_0^T \sigma^2 dt$.

Proof. For any nonnegative predictable R-indexed process H, we have

$$\begin{aligned}
E^*\Big(\int_0^T \int_R H(s,y)\nu(dy : ds)\Big) &= E\Big(L_T \int_0^T \int_R H(s,y)\nu(dy : ds)\Big) \\
&= E\Big(\int_0^T \int_R L_{s-} H(s,y) h(s,y) \nu(dy : ds)\Big) \\
&= E\Big(\int_0^T \int_R L_s H(s,y) h(s,y) \lambda_s H_s(dy) ds\Big) \\
&= E\Big(L_T \int_0^T \int_R H(s,y) h(s,y) \lambda_s H_s(dy) ds\Big) \\
&= E^*\Big(\int_0^T \int_R H(s,y) \lambda_s h(s,y) H_s(dy) ds\Big).
\end{aligned}$$

The first and second equality above follow from the fact that $L_T = dP^*/dP$ and $L_{\tau_n} = L_{\tau_n -} h(\tau_n, y_n)$ [2, A2, T19]. Hence the local characteristic of $\nu(dy : dt)$ is $(\lambda_t, h(t,y)H_t(dy))$. By Lemma 1 and Doleans-Dade exponential formula ([3]),

$$S(t) = S(0)\exp\left\{\int_0^t [(\mu(s,\omega) - \frac{1}{2}\sigma^2(s,\omega) + \sigma(s,\omega)dB(s) + \int_R \ln(1+\gamma(s:y))\nu(dy:ds)]\right\}$$

If we denote the sample continuous and pure jump part by S^c and S^d, respectively, we have $S^*(t) = S^{*c}(t)\cdot S^d(t)$ and $[S^{*c}, S^d] = 0$ by Hilbert space orthogonality. Hence

$$\begin{aligned} S^{*c} &= S(0)\exp\Big(\int_0^t (\mu - r - \frac{1}{2}\sigma^2)ds + \int_0^t \sigma dB\Big) \\ &= S(0)\exp\Big(\int_0^t \sigma dB^* - \frac{1}{2}\int_0^t \sigma^2 ds\Big) \end{aligned}$$

where

$$B^*(t) = B(t) + \int_0^t \frac{\mu - r}{\sigma} ds$$

is a $(P^*, \mathcal{F}_t)$-Brownian motion by Girsanov's Theorem ([2]). Therefore the discounted process $S^*(t)$ can be written by

$$S^*(t) = S(0)\exp\Big(\int_0^t \sigma dB^* - \frac{1}{2}\rho_t^2\Big)\prod_{n=1}^{N_t}(1+\gamma(\tau_n : Y_n)\Big)$$

where the jumps take place at random time τ_n with corresponding sizes Y_n. Here $\int_0^T \sigma dB^*$ follows Normal distribution with mean 0, variance ρ_T^2 under P^*. Therefore, (3) follows by first conditioning on $N(t) = k$, $Y_1 = y_1, \cdots, Y_k = y_k$, and then integrating $\psi(S_t) = \psi(S_t^* e^{\int_0^T r ds})$ with respect to the normal distribution $N(0, \rho_T^2)$.

We conclude with:

COROLLARY 3. *Consider the European call option $\psi(X) = X - c$, where c is the exercise price. Suppose that the riskless interest rate r is a constant P-a.s. Then the price at time zero π_T^c of the call option expiring at time T is given as*

$$\pi_T^c = e^{-rT} \sum_{k=0}^{\infty} e^{-\int_0^t \lambda_s ds} \frac{(\int_0^t \lambda_s ds)^k}{k!} \int \cdots$$
$$\int_R v(S(0) \prod_{n=1}^{k} (1 + \gamma(\tau_n, y_n)), T, c, \rho_T, r)$$
$$H_t^*(dy_1) \times \cdots \times H_t^*(dy_k)$$

where the function v is given as

$$v(S(0), T, c, \rho_T, r)$$
$$= S(0)\Phi\left(\frac{ln\frac{S(0)}{c} + rT + \frac{1}{2}\rho_T^2}{\rho_T}\right) - ce^{-rT}\Phi\left(\frac{ln\frac{S(0)}{c} + rT - \frac{1}{2}\rho_T^2}{\rho_T}\right)$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

Proof. Let S_T^c be the continuous part of S_T. Together with the definition of call option, we do this as follows:

$$e^{-rT}E^*(S_T^c - c|\mathcal{F}_0) = E^*(S_T^{*c} - ce^{-rT}|\mathcal{F}_0)$$
$$= E^*\left(S(0)\exp\left\{\int_0^T \sigma dB^* - \frac{1}{2}\int_0^T \sigma^2 dt\right\} - ce^{-rT}|\mathcal{F}_0\right)$$
$$= \int_{-\infty}^{\infty} \left(S(0)e^{y-\rho_T^2/2} - e^{-rT}c\right) \frac{e^{-y^2/2\rho_T^2}}{\sqrt{2\pi}\rho_T} dy$$
$$= \int_n^{\infty} \left(S(0)e^{y-\frac{1}{2}\rho_T^2} - ce^{-rT}\right) \frac{e^{-y^2/2\rho_T^2}}{\sqrt{2\pi}\rho_T} dy$$

$$= S(0)\int_n^\infty \frac{1}{\sqrt{2\pi}\rho_T} e^{-(y-\rho_T^2)^2/2\rho_T^2} dy - ce^{-rT}\int_n^\infty \frac{e^{-y^2/2\rho_T^2}}{\sqrt{2\pi}\rho_T} dy$$
$$= S(0)\Big[1 - \Phi(\frac{n-\rho_T^2}{\rho_T})\Big] - ce^{-rT}\Big[1 - \Phi(\frac{n}{\rho_T})\Big]$$
$$= S(0)\Phi\Big(\frac{\rho_T^2 - n}{\rho_T}\Big) - ce^{-rT}\Phi\Big(-\frac{n}{\rho_T}\Big)$$

where the third equality follows from $n = ln(\frac{c}{S(0)}) + \frac{1}{2}\rho_T^2 - rT$. Now we insert the expressions for n, then the fact that v has the indicated form follows from Theorem 2.

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