Bull. Korean Math. Soc. **33** (1996), No. 2, pp. 253–258

NILPOTENT ACTION BY AN ELEMENTARY AMENABLE GROUP AND EULER CHARACTERISTIC

JONG BUM LEE AND CHAN-YOUNG PARK

Let X be a finite connected CW-complex, $\Gamma = \pi_1(X)$ its fundamental group, $\widetilde{X}$ its universal covering space. Then Γ acts on $\widetilde{X}$ by covering transformations and on the homology group $H_*(\widetilde{X})$. In this note we establish the following vanishing result for the Euler characteristic $\chi(X)$ of X.

THEOREM. *If $\Gamma = \pi_1(X)$ contains a nontrivial, finite index, torsion free, elementary amenable subgroup A (i.e., Γ is elementary amenable and virtually torsion-free) which acts nilpotently on $H_*(\widetilde{X})$, then the Euler characteristic $\chi(X)$ of X vanishes.*

In the Theorem above, A acts nilpotently on $M = H_*(\widetilde{X})$ means that there exists a finite filtration $0 = M^{(0)} \subset M^{(1)} \subset \cdots \subset M^{(k-1)} \subset M^{(k)} = M$ by $\mathbb{Z}A$-modules such that A acts trivially on the associated graded module $\mathrm{Gr}M = \{M^{(i)}/M^{(i-1)} \mid i = 1, \cdots, k\}$. If X is finite and aspherical, then Γ is torsion-free and obviously Γ, and so any subgroup of Γ, acts nilpotently on $H_*(\widetilde{X})$. When X is finite and aspherical: Gottlieb [4] and Stallings [11] showed that if Γ has the nontrivial center then $\chi(X) = 0$; Rosset [10] showed that if Γ contains a nontrivial normal Abelian subgroup then $\chi(X) = 0$; Hillman and Linnell [5] showed that if Γ contains a nontrivial normal elementary amenable subgroup then $\chi(X) = 0$; Cheeger, Gromov [1] and Eckmann [3] showed that if Γ is an amenable group then $\chi(X) = 0$. The Theorem above is a variation of Eckmann's result [2] which states that if X is finite and if Γ contains a

Received December 10, 1994. Revised September 20, 1995.

1991 AMS Subject Classification: 57S25, 16S90, 20C07.

Key words and phrases: Nilpotent group action, elementary amenable group, Euler characteristic.

Supported in part by the Basic Science Research Institute Program, Ministry of Education, 1995, Project No. BSRI-95-1422 and BSRI-95-1409, and by TGRC-KOSEF.

nontrivial torsion-free normal Abelian subgroup which acts nilpotently on $H_*(\widetilde{X})$ then $\chi(X) = 0$. From the history of vanishing results for Euler characteristic, we can propose naturally the following problem.

QUESTION. *If X is a finite complex and if $\pi_1(X)$ contains a nontrivial torsion-free normal (elementary) amenable subgroup which acts nilpotently on $H_*(\widetilde{X})$ then $\chi(X)$ vanishes.*

The proof of the Theorem above is based on main results in [6] for group rings: since A is a torsion-free elementary amenable group, $\mathbb{C}A$ has a left quotient ring $D = (\mathbb{C}A \setminus \{0\})^{-1}\mathbb{C}A$. Using known results in the theory of rings of operators concerning von Neumann algebras we show that D has the "rank invariant" property for finitely generated free D-modules, i.e., all bases of such a module F consist of the same number of elements; this number is the *rank* of F over D, written $\mathrm{rk}_D F$. Using the assumption that A acts nilpotently on $H_*(\widetilde{X})$ we show also that $D \otimes_{\mathbb{C}A} H_*(\widetilde{X};\mathbb{C}) = 0$. If X is a finite CW-complex and if A is a finite index subgroup of $\pi_1(X)$ it follows that $\chi(X) = 0$.

By Theorem 1.4 of [6], $\mathbb{C}A$ is a domain; any nonzero element of $\mathbb{C}A$ is not a zero-divisor in $\mathbb{C}A$. By Theorem 1.3 of [6], $\mathbb{C}A$ has a left quotient ring which is a division ring D, i.e., $\mathbb{C}A$ is a subring of D, every nonzero element of $\mathbb{C}A$ is a unit in D, and every element of D is of the form $\alpha^{-1}\beta$ for some elements α and β of $\mathbb{C}A$ with $\alpha \neq 0$. Hence $D = (\mathbb{C}A \setminus \{0\})^{-1}\mathbb{C}A$ and D is a right flat $\mathbb{C}A$-module.

PROPOSITION 1. *D has the rank invariant property for finitely generated free D-modules.*

Proof. Since D is a division ring, every D-module is, by definition, a vector space over D. It is well-known that any two bases of a vector space over a division ring have the same cardinality, and every vector space over a division ring is a free module. Hence D has the rank invariant property for finitely generated D-modules. $\square$

REMARK 2. From the fact that D is a localized ring $(\mathbb{C}A \setminus \{0\})^{-1}\mathbb{C}A$ and using known results in the theory of rings of operators concerning von Neumann algebras we can give an alternate proof that D has the rank invariant property.

By (3.5) of [10], we can embed D in a von Neumann regular ring $\widetilde{W}(A)$ in such a way that

(a) $\widetilde{W}(A)$ is a ring of (densely defined) operators on the base the Hilbert space $\mathcal{H} := \ell_2(A)$,
(b) $\widetilde{W}(A)$ has the rank invariant property, and
(c) if $\mathcal{I}$ is a principal ideal of $\widetilde{W}(A)$ then $\mathcal{I}$ is generated by an idempotent element of the von Neumann algebra $W(A)$ generated by A.

By the universal property of localization it is enough to show that if $\alpha \in \mathbb{C}A$, $\alpha \neq 0$ then α is invertible in $\widetilde{W}(A)$. Define the operators L_α and R_α, respectively, by $L_\alpha(h) = \alpha h$ and $R_\alpha(h) = h\alpha$ for all $h \in \mathcal{H}$. Let $\widetilde{W}(G)\cdot L_\alpha = \widetilde{W}(G)\cdot e$ for some idempotent $e \in W(G)$. Suppose $e \neq \mathrm{id}_{\mathcal{H}}$. Then there is $h \in \mathcal{H}$ such that $(\mathrm{id}_{\mathcal{H}} - e)(h) \neq 0$. Since $L_\alpha \in \widetilde{W}(G)\cdot e$, $L_\alpha = w \circ e$ for some $w \in \widetilde{W}(G)$, and hence $L_\alpha \circ (\mathrm{id}_{\mathcal{H}} - e) = w \circ e \circ (\mathrm{id}_{\mathcal{H}} - e) = w \circ (e - e^2) = 0$. However by Theorem 2 of [8], $L_\alpha \circ (\mathrm{id}_{\mathcal{H}} - e)(h) \neq 0$, which is impossible. Thus $e = \mathrm{id}_{\mathcal{H}}$, $\widetilde{W}(G) = \widetilde{W}(G)\cdot L_\alpha$, and L_α has a left inverse in $\widetilde{W}(G)$. Similarly R_α has a right inverse in $\widetilde{W}(G)$. Thus D is embedded in the ring $\widetilde{W}(G)$ which has the rank invariant property. Hence D has the rank invariant property.

PROPOSITION 3. *If N is a trivial $\mathbb{C}A$-module then $D \otimes_{\mathbb{C}A} N = 0$.*

Proof. Indeed, there is an element $a \neq 1$ in A; for any $n \in N$ and $d \in D$ we have

$$d \otimes n = d(a-1)^{-1} \otimes (a-1)n = 0. \quad \square$$

PROPOSITION 4. *The localized homology $D \otimes_{\mathbb{C}A} H_*(\widetilde{X}; \mathbb{C})$ is 0.*

Proof. We write $M = H_*(\widetilde{X}; \mathbb{C})$, a $\mathbb{C}\Gamma$-module and hence a $\mathbb{C}A$-module, and consider a filtration of M given by the nilpotent action of A so that A acts nilpotently on the associated graded module $\mathrm{Gr}M$. By Proposition 3, $D \otimes_{\mathbb{C}A} \mathrm{Gr}M = 0$. Since D is a flat $\mathbb{C}A$-module this means $\mathrm{Gr}(D \otimes_{\mathbb{C}A} M) = 0$ for the corresponding filtration of $D \otimes_{\mathbb{C}A} M$. The filtration beginning with 0 it follows that $D \otimes_{\mathbb{C}A} M = 0$. $\square$

Proof of Theorem. Let $\underline{C} = \{0 \to C_n \to \cdots \to C_1 \to C_0 \to 0\}$ be the cellular chain complex over $\mathbb{C}$ of $\widetilde{X}$. It is a complex of free $\mathbb{C}\Gamma$-modules with $\mathrm{rk}_{\mathbb{C}\Gamma} C_i = \alpha_i$ = number of i-cells of X. (Note that $\mathbb{C}\Gamma$ has the rank invariant property, cf. [9].) Hence $\underline{C}$ is a complex of free $\mathbb{C}A$-modules with $\mathrm{rk}_{\mathbb{C}A} C_i = |\Gamma : A| \cdot \alpha_i$. From Proposition 4 and using again the flatness of D, we see that

$$0 = D \otimes_{\mathbb{C}A} H_*(\widetilde{X};\mathbb{C}) = D \otimes_{\mathbb{C}A} H_*(\underline{C}) = H_*(D \otimes_{\mathbb{C}A} \underline{C}).$$

Thus $0 \to D \otimes_{\mathbb{C}A} C_n \to \cdots \to D \otimes_{\mathbb{C}A} C_1 \to D \otimes_{\mathbb{C}A} C_0 \to 0$ is an exact sequence of finitely generated free D-modules. This implies

$$(D \otimes_{\mathbb{C}A} C_0) \oplus (D \otimes_{\mathbb{C}A} C_2) \oplus \cdots \quad \cong \quad (D \otimes_{\mathbb{C}A} C_1) \oplus (D \otimes_{\mathbb{C}A} C_3) \oplus \cdots$$

and, since D has the rank invariant property, we have

$$0 = \sum_{i=0}^{n} (-1)^i \mathrm{rk}_D (D \otimes_{\mathbb{C}A} C_i) = \sum_{i=0}^{n} (-1)^i |\Gamma : A| \cdot \alpha_i \ \text{ or } \ \sum_{i=0}^{n} (-1)^i \alpha_i = 0.$$

I.e., the Euler characteristic $\chi(X)$ is 0, which proves the Theorem. □

The *cohomological dimension* of a discrete group G, denoted $\mathrm{cd}(G)$, is defined by

$$\begin{aligned} \mathrm{cd}(G) &= \inf\{n \mid \mathbb{Z} \text{ admits a projective resolution of length } n\} \\ &= \inf\{n \mid H^i(G,-)) = 0 \text{ for } i > n\} \\ &= \sup\{n \mid H^n(G,A) \neq 0 \text{ for some } G\text{-module } A\}. \end{aligned}$$

A group G is said to be of type *FP* if $\mathbb{Z}$ admits a resolution

$$0 \to P_n \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

of finite length such that the P_i are finitely generated projective $\mathbb{Z}G$-modules. We say that a group *virtually* has a given property if some subgroup of finite index has that property. For example, $\mathrm{vcd}(G) = \mathrm{cd}(G_0)$ for some subgroup G_0 of G with finite index. If G is virtually of type *FP* and G_0 is any finite index subgroup of G, then the Euler characteristic $\chi(G)$ of G is defined by

$$\chi(G) = \frac{\chi(G_0)}{|G : G_0|},$$

where $\chi(G_0) = \sum (-1)^i \mathrm{rk}_{\mathbb{Z}} H^i(G_0, \mathbb{Z})$.

COROLLARY. *Let X be a finite connected CW-complex, $\Gamma = \pi_1(X)$ an infinite elementary amenable group with finite virtual cohomological dimension, $\widetilde{X}$ homotopic to an even-dimensional sphere S^{2k}. Then Γ is virtually of type FP and $\chi(\Gamma) = 0$.*

Proof. Recall that a nontrivial finite subgroup of $\Gamma = \pi_1(X)$ is isomorphic to $\mathbb{Z}_2$, Γ is torsion-free or else Γ is isomorphic to $\Gamma' \rtimes \mathbb{Z}_2$ where Γ' is torsion-free. If $\mathrm{vcd}(\Gamma) < \infty$ then Γ is virtually of type *FP*, and so $\chi(\Gamma)$ is defined and $\chi(X) = \chi(\Gamma) \cdot \chi(\widetilde{X})$ ([7]). Note also that since $H_{2k}(\widetilde{X}) = \mathbb{Z}$, the kernel of the induced action homomorphism $\Gamma \to \mathrm{Aut}(H_{2k}(\widetilde{X})) = \mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$ has index at most 2 in Γ and acts nilpotently on $H_*(\widetilde{X})$. Hence Γ has a nontrivial torsion-free finite index elementary amenable subgroup which acts nilpotently on $H_*(\widetilde{X})$. By Theorem and by the fact that $\chi(\widetilde{X}) = 2$, we have $0 = \chi(X) = 2 \cdot \chi(\Gamma)$. Thus $\chi(\Gamma) = 0$. □

References

1. J. Cheeger and M. Gromov, *L_2-cohomology and group cohomology*, Topology **25** (1986), 189-215.
2. B. Eckmann, *Nilpotent group action and Euler characteristic*, Lecture Note in Mathematics **1248** (1985), 120-123.
3. B. Eckmann, *Amenable groups and Euler characteristic*, Comment. Math. Helvetici **67** (1992), 383-393.
4. D. H. Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. **87** (1965), 840-856.
5. J. A. Hillman and P. A. Linnell, *Elementary amenable groups of finite Hirsch length are locally-finite by virtually solvable*, J. Austral. Math. Soc. **52** (1992), 237-241.
6. P. H. Kropholler, P. A. Linnell and J. A. Moody, *Applications of a new K-theoretic theorem to soluble group rings*, Proc. Amer. Math. Soc. **104** (1988), 675-684.
7. J. B. Lee, *Transformation groups on $S^n \times \mathbb{R}^m$*, Topology and its Applications **53** (1993), 187-204.
8. P. A. Linnell, *Zero divisors and group von Neumann algebras*, Pacific J. Math. **149** (1991), 349-363.
9. S. M. Montgomery, *Left and right inverses in group algebras*, Bull. Amer. Math. Soc. **75** (1969), 539-540.
10. S. Rosset, *A vanishing theorem for Euler characteristics*, Math. Z. **185** (1984), 211-215.
11. J. R. Stallings, *Centerless groups-an algebraic formulation of Gottlieb's theorem*, Topology **4** (1965), 129-134.

JONG BUM LEE
DEPARTMENT OF MATHEMATICS, SOGANG UNIVERSITY, SEOUL 121-742, KOREA
E-MAIL: JLEE@CCS.SOGANG.AC.KR

CHAN-YOUNG PARK
DEPARTMENT OF MATHEMATICS, KYUNGPOOK NATIONAL UNIVERSITY, TAEGU 702-701, KOREA
E-MAIL: CHNYPARK@BH.KYUNGPOOK.AC.KR